Three Dollars per Annum, in Advance. Single Copies, Twenty-Five Cents.

VOL. III. No. XII.

THE

MATHEMATICAL MONTHLY.

SEPTEMBER, 1861.

EDITED BY

J. D. RUNKLE, A.M., A.A.S.

CAMBRIDGE:
PUBLISHED BY SEVER AND FRANCIS.
NEW YORK:
A. S. BARNES AND BURR.
LONDON: TRÜBNER & CO.
1861.

ICHNOGRAPHS

FROM THE

CONNECTICUT-RIVER SANDSTONE.

BY

JAMES DEANE, M. D.

JUST PUBLISHED

BY LITTLE, BROWN, & CO., BOSTON.

4to. PRICE, $5.

THIS work consists of a series of forty-six Plates, containing about eighty figures, of the Fossil Footprints found on the Sandstone of the Connecticut River, — prepared under the direction of the late Dr. JAMES DEANE, of Greenfield, who for many years, until his death, was almost constantly engaged in their investigation. Nine of the plates are photographs, and the remainder are very beautifully executed on stone; and, rivalling as they do the actual specimens, they will be of great use to those pursuing scientific investigations in this direction; or, at least, they will furnish a beautiful table book, to excite an interest in the community in the marvels of nature.

A Memoir of the Footprints and other impressions, written by Dr. Deane, is given, together with a particular description of each plate, and a Biographical Notice of the Author, — the whole being comprised in an elegant quarto volume.

The work was prepared for publication by THOMAS T. BOUVÉ, Esq., Dr. H. I. BOWDITCH, and Dr. A. A. GOULD.

Only 300 copies of this work are printed, most of which are already subscribed for.

THE MATHEMATICAL MONTHLY.

The excellent notices which the Monthly continues to receive from both sides of the Atlantic must be a source of great encouragement to all its friends. The *Lady's and Gentleman's Diary* for 1861, edited by the eminent mathematician, PROF. W. S. B. WOOLHOUSE, contains the following: —

"This American periodical, which has now completed its second volume, continues to be most ably conducted, and is well supplied by talented correspondents. Besides numerous mathematical questions and solutions, it comprises original papers of considerable variety and sterling interest, and its publication must contribute largely to the diffusion and advancement of mathematical science."

A recent number of the *Boston Daily Transcript* gives the following item: —

"AMERICAN LITERATURE ABROAD. — Fourteen columns of the last Athenæum are devoted to two American authors, — Motley and Emerson. That paper, which rarely commends anything from this side of the Atlantic, gives 'The Mathematical Monthly' the following first-rate notice. That magazine is published at Cambridge, and edited by Mr. J. D Runkle: —

"'This is a collection of simple things for comparative elementary students. Our English mathematical journals are very lofty, and the beginner has no part in them. This American journal is full of pleasing elementary matter, and contains some things worthy the attention of the finished mathematician. We should like to see the same sort of thing in England; but where are we to find the editor? He must be of high knowledge and elementary turn combined, with firmness enough to keep his journal down to the intelligible point by refusal of contributions of too lofty a character, and skill enough to keep it up to the interesting point by selecting from the loads of lower learning with which he would be inundated.'"

Those wishing India-proof impressions of SIR JOHN HERSCHEL'S or PROF. PEIRCE'S Portrait, will please address the Publishers.

Those in arrears are earnestly requested to remit their subscriptions, if possible, before the issue of the next number.

SEVER & FRANCIS, PUBLISHERS,

Cambridge, Mass.

Contents.

SEPTEMBER, 1861.

	PAGE
ON THE RESULTS OF ACCELERATED VELOCITY. By John Paterson	357
FOURTH PRIZE ESSAY. — SPHERICAL CONICS. By C. M. Woodward	361
EQUATION AND CONSTRUCTION OF THE HYPERBOLA. By John Warner	375
CONVENIENT FORMULÆ FOR INTERPOLATION. By William Ferrel	377

Terms to Subscribers and Advertisers.

No. of Copies to one Address.	Volume.	Terms.	Premium to Clubs.	Advertisement Pages.	First Insertion.	Subsequent Insertions.
1	III.	$3.00		1	$15.00	$12.00
2	III.	5.00		½	9.00	7.00
3	I., II., III.	6.00		¼	5.00	4.00
5	III.	11.00	III.	⅛	3.00	2.00
10	III.	20.00	II., III.			
15	III.	30.00	I., II., III.			

Parties in New York wishing to advertise in the Mathematical Monthly will please send their favors to Messrs. A. S. BARNES AND BURR by the 15th of each month. All club subscriptions and business communications must be addressed to

SEVER AND FRANCIS, Publishers,

Cambridge, Mass.

University Press, Cambridge: Electrotyped and Printed by Welch, Bigelow, & Co.

THE

MATHEMATICAL MONTHLY.

Vol. III. . . . SEPTEMBER, 1861. . . . No. XII.

ON THE RESULTS OF ACCELERATED VELOCITY.

By JOHN PATERSON, A. M., Albany, N. Y.

I. IF a material unit, 1_μ, move uniformly (*in vacuo*) in time, and 1_l be the distance it travels or describes in the assumed unit of time 1_t, it will describe the distance $x\,1_l$ in the time $x\,1_t$: the velocity is constant; its unit may be written 1_λ, which is its measure as a cause, and is equal to 1_l, the measure of its effect in the unit of time 1_t; and the distance the constant velocity generates increases uniformly as the time, being equal to the product of the velocity by the time $1_\lambda \,.\, x\,1_t = x\,1_l$.

II. If the velocity increase uniformly in time (beginning at zero), and $m\,1_\lambda$ be the amount accruing in the first unit of time 1_t (m to be determined), this amount will become $m\,x\,1_\lambda$ in the time $x\,1_t$. At the expiration of the time $x\,1_t$ let the increase cease, and the velocity remain constant with its ultimate value acquired in this time. With this ultimate constant value, the velocity will carry the material unit or body 1_μ some number, m, of times the distance in the succeeding time $x\,1_t$ that it carried the same body in the past equal time $x\,1_t$, while increasing from zero to its ultimate value, $m\,x\,1_\lambda$; that is, if l be the distance described or generated in the past time $x\,1_t$, $m\,l$ will be the distance generated in the future equal time $x\,1_t$, and, conversely, the former distance will be the mth part of the latter; and

it is obvious that l may be taken for unit of distance 1_l. Thus $m x 1_\lambda$ expresses the amount of velocity (now to remain constant) acquired by 1_μ in the first interval $x 1_t$ of time. Since $1_\lambda = 1_l$, this velocity will carry 1_μ the distance $m x 1_l$ in the succeeding unit of time 1_t, and the distance $m x^2 1_l$ in the time $x 1_t$: consequently the mth part of this distance, $= x^2 1_l$, is the distance described by 1_μ while the velocity was increasing uniformly from zero to the ultimate value $m x 1_\lambda$ in the past time $x 1_t$.

Divide the time into any two (equal or unequal) intervals, $\frac{1}{n} x 1_t$ and $\frac{n-1}{n} x 1_t$. As the final velocity has increased uniformly in time, it is $\frac{m}{n} x 1_\lambda$ and $\frac{(n-1)m}{n} x 1_\lambda$ at the expiration of these respective intervals of time. The velocity $\frac{m}{n} x 1_\lambda$ would carry 1_μ the distance $\frac{m}{n} x^2 1_l$ in the time $x 1_t$, which reduces to $\frac{m(n-1)}{n^2} x^2 1_l$ for the interval $\frac{n-1}{n} x 1_t$ (being the product of the velocity by the time), and to $\frac{1}{n^2} x^2 1_l$ for the past interval $\frac{1}{n} x 1_t$ (being the mth part of the product of the ultimate velocity by the time). The velocity $\frac{m(n-1)}{n} x 1_\lambda$ would carry 1_μ the distance $\frac{m(n-1)}{n} x^2 1_l$ in the time $x 1_t$, which reduces to $\frac{m(n-1)^2}{n^2} x^2 1_l$ for the subsequent interval $\frac{n-1}{n} x 1_t$, and to $\left(\frac{n-1}{n}\right)^2 x^2 1_l$ for the interval $\frac{n-1}{n} x 1_t$ during which the ultimate value was acquired. Since $x^2 1_l$ is the whole distance described by 1_μ in the entire interval $x 1_t$ of past time, there has arisen the equation

$$\left(\frac{1}{n^2} x^2 + \frac{m(n-1)}{n^2} x^2 + \left(\frac{n-1}{n}\right)^2 x^2\right) 1_l = x^2 1_l;$$

whence $m = 2$.

Hence, when a velocity increases uniformly, its ultimate value for the time $x 1_t$ is $2 x 1_\lambda$, and $x^2 1_l$ is the distance it has generated while acquiring this ultimate value; that is, when the velocity increases as the time, the distance increases as the square of the time; or, the

increase of the velocity is uniform, and the movement is uniformly accelerated.

III. Suppose, next, that the increase of the velocity is uniform, or that the velocity is uniformly accelerated, and therefore increases as the square of the time. If l be the distance generated in the past time $x\,1_t$, the ultimate velocity, become constant, will generate the distance $m'l$ in the equal future time $x\,1_t$ (m' to be determined). But $m'x^2\,1_\lambda$, being the ultimate velocity, will generate the distance $m'x^2\,1_t$ in the unit of time 1_t, and $m'x^3\,1_t$ in the time $x\,1_t$: consequently $x^3\,1_t$ was generated in the past time $x\,1_t$.

Dividing the time $x\,1_t$ into two intervals, as before, the entire right-hand member of the equation in the preceding paragraph is constituted of three partial terms in the left-hand member,

$$\frac{1}{n^2}x^2 + \frac{2(n-1)}{n^2}x^2 + \left(\frac{n-1}{n}\right)^2 x^2;$$

and now, instead of increments of distance, these several terms represent increments of velocity, each increasing from zero to an ultimate value, to be expressed by introducing the factor m'.

Now $\frac{m'}{n^2}x^2\,1_\lambda$ would generate $\frac{m'}{n^2}x^2\,1_t$ in the unit of time 1_t, and therefore $\frac{m'}{n^2}x^3\,1_t$ in the time $x\,1_t$, which reduces to $\frac{m'(n-1)}{n^3}x^3\,1_t$ for the time $\frac{n-1}{n}x\,1_t$ and to $\frac{1}{n^3}x^3\,1_t$ for the time $\frac{1}{n}x\,1_t$; and, similarly, $\frac{m'(n-1)^2}{n^2}x^2\,1_\lambda$ generates $\frac{(n-1)^3}{n^3}x^3\,1_t$ in the time $\frac{n-1}{n}x\,1_t$. Each of these increments of velocity is *uniformly accelerated* from zero to the ultimate value expressed by m'; but the increment $\frac{2m'(n-1)}{n^2}x^2\,1_\lambda$ only increases uniformly from zero to its ultimate value during the time $\frac{n-1}{n}x\,1_t$; and hence, while, with its ultimate value constant, it would generate the distance $\frac{2m'(n-1)}{n^2}x^3\,1_t$ in the time $x\,1_t$, this distance first reduces to $\frac{2m'(n-1)^2}{n^3}x^3\,1_t$ for the subsequent time $x\,1_t$, and then to $\frac{2}{2}\cdot\frac{m'(n-1)^2}{n^3}x^3\,1_t$ for the time $\frac{n-1}{n}x\,1_t$ during which the

increment acquired its ultimate value by uniform increase. Hence appears the equation

$$\left(\frac{1}{n^3}x^3+\frac{m'(n-1)}{n^3}x^3+\frac{m'(n-1)^2}{n^3}x^3+\frac{(n-1)^3}{n^3}x^3\right)1_l=x^3 1_l,$$

which gives $m'=3$. Consequently, when the velocity is uniformly accelerated, or increases in duplicate ratio of the time, its ultimate value is $3x^2 1_\lambda$; which, being constant, would carry 1_μ the distance $3x^3 1_l$ in the coming time $x 1_t$, one third of which is $1x^3 1_l$, the distance 1_μ was carried in the past equal time $x 1_t$: the distance increases as the cube of the time (in triplicate ratio).

IV. If the velocity be duplicately accelerated, or increase in triplicate ratio of the time, the preceding example prescribes the partial velocities

$$\frac{m''}{n^3}x^3 1_\lambda+\frac{3m''(n-1)}{n^3}x^3 1_\lambda+\frac{3m''(n-1)^2}{n^3}x^3 1_\lambda+\frac{m''(n-1)^3}{n^3}x^3 1_\lambda;$$

the first and last terms of which increase triplicately in the respective times $\frac{1}{n}x 1_t$ and $\frac{n-1}{n}x 1_t$, and, reasoning similarly as in the foregoing cases, their results are $\frac{1}{n^4}x^4 1_l$ in $\frac{1}{n}x 1_t$, and $\frac{m''(n-1)}{n^4}x^4 1_l$ and $\frac{(n-1)^4}{n^4}x^4 1_l$ in $\frac{n-1}{n}x 1_t$: the second term increases uniformly in the time $\frac{n-1}{n}x 1_t$, and therefore its result is $\frac{3}{2}\frac{m''(n-1)^2}{n^4}x^4 1_l$ in that time; and the third term increases duplicately, and its result is $\frac{3}{3}\frac{m''(n-1)^3}{n^4}x^4 1_l$ in the time $\frac{n-1}{n}x 1_t$. Arrangement and addition show

$$\begin{aligned}
&\frac{1}{n}x 1_t + \frac{n-1}{n}x 1_t = 1x 1_t.\\
&\left.\begin{aligned}
&\frac{1}{n^4}x^4 1_l+\frac{m''(n-1)}{n^4}x^4 1_l\\
&\quad+\frac{3}{2}\frac{m''(n-1)^2}{n^4}x^4 1_l\\
&\quad+\frac{3}{3}\frac{m''(n-1)^3}{n^4}x^4 1_l\\
&\quad+\frac{(n-1)^4}{n^4}x^4 1_l
\end{aligned}\right\}=1x^4 1_l,
\end{aligned}$$

which gives $m'' = 4$. The result increases as the biquadrate of the time.

The same method of investigation applies to cases where the velocity is triplicately, quadruplicately, etc. accelerated. The *a priori* or deductive method, however, gives at once the complete demonstration of the genesis of the entire series or hierarchy of corresponding simultaneous and successive powers (or faculties) denoted as follows: —

$$1\varphi^n,\ 1x\varphi^{n-1},\ 1x^2\varphi^{n-2},\ 1x^3\varphi^{n-3},\ \ldots\ 1x^{n-3}\varphi''',\ 1x^{n-2}\varphi'',\ 1x^{n-1}\varphi',\ 1x^n\varphi^\circ,$$

and

$$1\varphi^n,\ 2x\varphi^{n-1},\ 3x^2\varphi^{n-2},\ 4x^3\varphi^{n-3},\ \ldots\ (n-2)x^{n-3}\varphi''',\ (n-1)x^{n-2}\varphi'',\ nx^{n-1}\varphi',\ 1x^n\varphi^\circ,$$

which govern the entire theory of development, and exhibit the actual composition of the differential and integral coefficients and the origin of factorials. But inasmuch as this demonstration has heretofore been published, and I believe has failed to excite any attention, I have prepared the present article more in accordance with the methods of induction, and will be thankful for an exposure of its fallacy.

FOURTH PRIZE ESSAY.—SPHERICAL CONICS.

By C. M. Woodward, Newburyport, Mass.

1. It is the object of this paper to give some equations, and to investigate the nature, of the intersections made by a cone on the surface of a sphere whose centre is at the vertex. The cone may be any cone of the second degree; i. e. it may have any section of the second degree for its base.

Let C be the centre of the sphere, and O the foot of the perpendicular dropped from C to the base of the cone; and let $CO =$ the

radius of the sphere = unity, so that the base may be tangent to the sphere at O. Now, let O be the origin of a system of rectangular co-ordinates, and let

$$f(x, y) = 0 \tag{1}$$

be the equation of the base of the cone. The equations for transforming (1) to a system of polar co-ordinates, the origin being the same, and the axis of X being the polar axis, are $x = r\cos\varphi, y = r\sin\varphi$. If we let ϱ be the arc of the sphere corresponding to the radius vector r, so that $r = \tan\varrho$, these values of x and y become

$$\left.\begin{aligned} x &= \tan\varrho\ \cos\varphi, \\ y &= \tan\varrho\ \sin\varphi, \end{aligned}\right\} \tag{2}$$

which, substituted in (1), give

$$f(\tan\varrho\cos\varphi,\ \tan\varrho\sin\varphi) = 0. \tag{3}$$

Now, if the curve represented by the base of the cone be projected upon the sphere on right lines drawn towards its centre, it will evidently become the curve of the intersection of the cone and the sphere; i. e. it will be a spherical conic. But ϱ, the spherical radius vector, and φ, the polar angle, are the independent variables of the conic; and certain functions of these, derived from (2), must be substituted for x and y in the rectangular equations of the base, in order to give the equations of the spherical conic in terms of ϱ and φ.

2. It is evident, from the nature of the sphere and cone, that whatever may be the section of the cone made by its base, or, in other words, wherever we take the point O, at which point the base is tangent to the sphere, the projection of the plane curve always gives the *same spherical curve.* And it will be shown that, as the plane section of the cone is an ellipse, an hyperbola, or a parabola, according to the position of the point O, so the spherical curve, when viewed from the same positions, will appear to be in turn a spherical

ellipse, a spherical hyperbola, and a spherical parabola, though the curve itself remains in all cases the same.

3. *Equation of the Spherical Ellipse.* — Let us now take the general equation of a cone,

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = 0, \tag{4}$$

and find from it the equation of a section parallel to the plane xy, and at a distance from it equal to unity. Substituting $z = 1$ in (4), it becomes $\frac{C^2x^2}{A^2} + \frac{C^2y^2}{B^2} = 1$, and if we make $\frac{A}{C} = a$, $\frac{B}{C} = b$, whence

$$\frac{A}{B} = \frac{a}{b}, \tag{5}$$

it reduces to the form $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{6}$$

which is the equation of an ellipse, the origin being at the centre and the axes $2a$ and $2b$. Since the axis of z is the axis of the cone, the point O is the centre of the ellipse.

Equation (6) becomes, by substituting (2),

$$\frac{\tan^2\varrho\cos^2\varphi}{a^2} + \frac{\tan^2\varrho\sin^2\varphi}{b^2} = 1,$$

or,
$$\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2} = \frac{1}{\tan^2\varrho}. \tag{7}$$

Now, if $\varphi = 0°$, ϱ coincides with the transverse axis of the spherical curve, which we will call 2α, and (7) gives

$$\frac{1}{a^2} = \frac{1}{\tan^2\varrho}, \qquad \text{or,} \qquad a = \tan\varrho = \tan\alpha. \tag{8}$$

Also, if we make $2\beta =$ the conjugate axis when $\varphi = 90°$, (7) gives

$$\frac{1}{b^2} = \frac{1}{\tan^2\varrho}, \qquad \text{or,} \qquad b = \tan\varrho = \tan\beta. \tag{9}$$

Substituting these values of a and b in (7), we get

$$\frac{\cos^2\varphi}{\tan^2\alpha} + \frac{\sin^2\varphi}{\tan^2\beta} = \frac{1}{\tan^2\varrho}, \tag{10}$$

which is the equation of a spherical ellipse referred to its principal diameters.

4. *Discussion of* (10). — Since the first member of (10) is always positive, φ may have any value, and the curve extends completely around the origin. But as the cone has two nappes, its intersections with the sphere will give two equal spherical curves, which are diametrically opposite to each other; so that the equation, to be complete, must be the equation of both curves. Now the tangent of any angle is the same as the tangent of the angle plus 180°, or π; hence every value of $\tan \varrho$ gives two values of the radius vector differing from each other by a semi-circumference. Thus, when $\varphi = 0°$, (10) gives $\tan \varrho = \pm \tan \alpha$, $\varrho = \pm \alpha$, or $\pm (\pi + \alpha)$, determining two points on each curve. Therefore (10) is sufficient for both curves, even if we neglect the negative values of $\tan \varrho$.

5. *Equation of the Spherical Hyperbola.* — If we substitute in (4) the value $x = 1$, we shall get the equation of the section made by a plane parallel to the plane $y\,z$, and tangent to the sphere. Since this plane is perpendicular to the plane considered in § 3, O', the point of tangency to the sphere and the centre of the hyperbola, is 90° from the point O, which was the centre of the ellipse. In this case the resulting equation is $\frac{A^2 z^2}{C^2} - \frac{A^2 y^2}{B^2} = 1$, which by (5) becomes

$$a^2 z^2 - \frac{a^2}{b^2} y^2 = 1, \tag{11}$$

which is the equation of a plane hyperbola referred to its principal diameters. Substituting in (11) the values of y and z derived from (2), (for z can obviously take the place of x,) and representing in this case the radius vector and the polar angle by ϱ_1 and φ_1, we get

$$a^2 \tan^2 \varrho_1 \cos^2 \varphi_1 - \frac{a^2}{b^2} \tan^2 \varrho_1 \sin^2 \varphi_1 = 1,$$

$$a^2 \cos^2 \varphi_1 - \frac{a^2}{b^2} \sin^2 \varphi_1 = \frac{1}{\tan^2 \varrho_1}. \tag{12}$$

Now if we denote the semi-axes of this curve by α_1 and β_1, when $\varphi_1 = 0°$, (12) gives $a^2 = \frac{1}{\tan^2 \varrho_1}$, or $a = \cot \varrho_1 = \cot \alpha_1 = \tan \alpha$; thus showing that α and α_1 are complements of each other. Since the semi-conjugate axis of the plane hyperbola is $\frac{b}{a}$, we shall have $\tan \beta_1 = \frac{b}{a} = \frac{\tan \beta}{\tan \alpha}$. Hence we get from (12) the equation

$$\frac{\cos^2 \varphi_1}{\tan^2 \alpha_1} - \frac{\sin^2 \varphi_1}{\tan^2 \beta_1} = \frac{1}{\tan^2 \varrho_1}, \qquad (13)$$

which is the equation of a spherical hyperbola, the origin being at the centre, and the axes being $2\alpha_1$ and $2\beta_1$.

6. *Discussion of* (13). — It is evident from the nature of the equation that ϱ_1 will be real for every value of φ_1, which makes $\frac{\cos^2 \varphi_1}{\tan^2 \alpha_1} > \frac{\sin^2 \varphi_1}{\tan^2 \beta_1}$, and imaginary for all other values.

The limits to the value of φ_1 are determined by the equation $\frac{\cos^2 \varphi_1}{\tan^2 \alpha_1} = \frac{\sin^2 \varphi_1}{\tan^2 \beta_1}$, which gives

$$\tan^2 \varphi_1 = \frac{\tan^2 \beta_1}{\tan^2 \alpha_1} = \tan^2 \beta,$$

or, $\tan \varphi_1 = \pm \tan \beta$, $\varphi_1 = \pm \beta$, or, $\pm(\pi + \beta)$.

Hence it is plain that for any value of φ_1 between β and $\pi - \beta$, and between $-\beta$ and $-(\pi - \beta)$, we shall have $\tan^2 \varphi_1 > \tan^2 \beta$, or $\frac{\cos^2 \varphi_1}{\tan^2 \alpha_1} < \frac{\sin^2 \varphi_1}{\tan^2 \beta_1}$; so that ϱ_1 will be real only for those values of φ_1 between β and $-\beta$, and between $\pi - \beta$ and $\pi + \beta$. When $\varphi_1 = 0°$, (13) gives

$$\tan \varrho_1 = \pm \tan \alpha_1 \quad \text{and} \quad \varrho_1 = \pm \alpha_1, \quad \text{or,} \quad = \pm(\pi + \alpha_1),$$

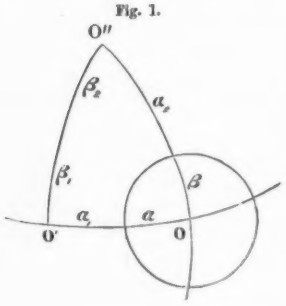
so that every value of φ_1 for which ϱ_1 is real gives four points, two on each branch of the curve. Hence, as in the ellipse, the negative sign may be neglected. It is evident that this curve lacks one of the essential characteristics of the plane hyperbola, viz. its infinite branches; and that the curve on each branch, on reaching a point 90° from the origin, returns symmetrically into itself.

7. *Equation of Spherical Hyperbola referred to another Set of Axes.* — If we take a section of the cone parallel to the plane xz, letting $y = 1$, we shall get in a similar manner the equation

$$b^2 z^2 - \frac{b^2}{a^2} x^2 = 1, \tag{14}$$

which is the equation of an hyperbola lying in a plane perpendicular to that expressed by (13). Since O', the point of tangency in this case, is the pole of the arc OO', the transverse axis α_2 of the curve is the complement of the conjugate axis β of the ellipse, and the conjugate axis β_2 is the complement of β_1 of the other hyperbola, as is evident from Fig. 1.

Fig. 1.



Hence we have $\tan \alpha_2 = \cot \beta = \frac{1}{b}$, and $\tan \beta_2 = \cot \beta_1 = \frac{\tan \alpha}{\tan \beta} = \frac{a}{b}$. The substitution of these values in (14), together with the values of z and x derived from (2) by putting z for x and x for y, and using ϱ_2 and φ_2, gives

$$\frac{\cos^2 \varphi_2}{\tan^2 \alpha_2} - \frac{\sin^2 \varphi_2}{\tan^2 \beta_2} = \frac{1}{\tan^2 \varrho_2}, \tag{15}$$

which is the equation of a spherical hyperbola (Fig. 1), its centre being the origin, and its axes being $2\alpha_2$ and $2\beta_2$. The discussion of (15) is similar to that of § 6.

8. *Equation of Spherical Parabola.* — Let us now transform equation (4) to a new system of rectangular co-ordinates, whose axes are X_1, Y_1, and Z_1, having the same origin, the axis of Y_1 coinciding with Y, and the axes of X_1 and Z_1 making the angle α with X and Z respectively. It is evident, since α is half the angle at the vertex of the cone in the plane xz, that the new axis Z_1 coincides with the side of the cone, and the section made by a plane parallel to $y_1 z_1$ will be a

parabola. On these conditions the equations for transformation are $x = x_1 \cos \alpha - z_1 \sin \alpha$, $y = y_1$, $z = x_1 \sin \alpha + z_1 \cos \alpha$. Substituting these values in (4), it becomes

$$\frac{x_1^2 \cos^2\alpha - 2 x_1 z_1 \cos\alpha \sin\alpha + z_1^2 \sin^2\alpha}{A^2} + \frac{y_1^2}{B^2} - \frac{x_1^2 \sin^2\alpha + 2 x_1 z_1 \sin\alpha \cos\alpha + z_1^2 \cos^2\alpha}{C^2} = 0,$$

$$y_1^2 = \frac{B^2}{C^2}(x_1^2 \sin^2\alpha + 2 x_1 z_1 \sin\alpha \cos\alpha + z_1^2 \cos^2\alpha) - \frac{B^2}{A^2}(x_1^2 \cos^2\alpha - 2 x_1 z_1 \sin\alpha \cos\alpha + z_1^2 \sin^2\alpha).$$

If we make $x_1 = 1$, the above equation becomes the equation of a section made by a plane parallel to $y_1 z_1$, and tangent to the sphere. This substitution gives

$$y_1^2 = \frac{B^2}{C^2}(\sin^2\alpha + z_1 \sin 2\alpha + z_1^2 \cos^2\alpha) - \frac{B^2}{A^2}(\cos^2\alpha - z_1 \sin 2\alpha + z_1^2 \sin^2\alpha),$$

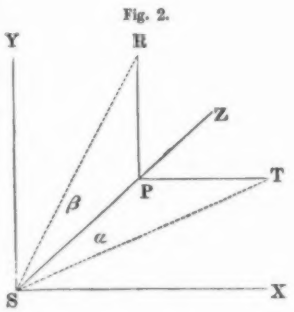
$$= \frac{B^2}{C^2}\sin^2\alpha - \frac{B^2}{A^2}\cos^2\alpha + \sin 2\alpha \left(\frac{B^2}{C^2} + \frac{B^2}{A^2}\right) z_1 + \left(\frac{B^2}{C^2}\cos^2\alpha - \frac{B^2}{A^2}\sin^2\alpha\right) z_1^2. \quad (16)$$

Let P (Fig. 2) be any point in the axis of Z, and draw PT parallel to SX, and PR parallel to SY, so that T and R may be at the surface of the cone. Then the lines PT, PR, and SP represent the ratios which the quantities A, B, and C bear to each other in the equation of the cone. Also, the angle $TSP = \alpha$ and $RSP = \beta$. Hence $\cos\alpha = \frac{SP}{ST} = \frac{SP}{\sqrt{((SP)^2 + (PT)^2)}}$,

$$\cos^2\alpha = \frac{SP^2}{SP^2 + PT^2} = \frac{C^2}{C^2 + A^2},$$

also, $$\sin^2\alpha = \frac{A^2}{C^2 + A^2}.$$

Fig. 2.

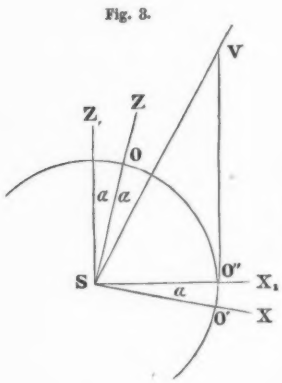


Substituting these values in the coefficient of z_1^2 in (16), it becomes $\frac{B^2}{C^2} \times \frac{C^2}{A^2 + C^2} - \frac{B^2}{A^2} \times \frac{A^2}{A^2 + C^2} = 0$; so that the term containing z_1^2 disappears, and (16) becomes

$$y_1^2 = \frac{B^2}{C^2}\sin^2\alpha - \frac{B^2}{A^2}\cos^2\alpha + \sin 2\alpha \left(\frac{B^2}{C^2} + \frac{B^2}{A^2}\right) z_1, \quad (17)$$

which is the rectangular equation of a parabola lying in the plane $y_1 z_1$, the axis of z_1 being the axis of the curve, and the distance of the vertex from the origin being

Fig. 3.



$$\frac{\frac{B^2}{A^2}\cos^2\alpha - \frac{B^2}{C^2}\sin^2\alpha}{\sin 2\alpha\left(\frac{B^2}{C^2}+\frac{B^2}{A^2}\right)}. \qquad (18)$$

Let O'' (Fig. 3) be the point at which the cutting plane is tangent to the sphere, and let V be the vertex of the parabola. Then obviously

$$O''V = \tan(\tfrac{1}{2}\pi - 2\alpha) = \cot 2\alpha = (18).$$

This last equation is shown to be true by reducing (18) and $\cot 2\alpha$ to the same form. Thus, by substituting values for $\cos^2\alpha$ and $\sin^2\alpha$, (18) becomes

$$\frac{\frac{B^2}{A^2}\left(\frac{C^2}{A^2+C^2}\right)-\frac{B^2}{C^2}\left(\frac{A^2}{A^2+C^2}\right)}{\left(\frac{B^2}{C^2}+\frac{B^2}{A^2}\right)\frac{2AC}{A^2+C^2}} = \frac{\frac{C^2}{A^2}-\frac{A^2}{C^2}}{\frac{2A}{C}+\frac{2C}{A}} = \frac{\frac{C^4-A^4}{AC}}{2(A^2+C^2)} = \frac{C^2-A^2}{2AC}. \qquad (19)$$

Also, from (5) and (8) we have

$$\cot 2\alpha = \frac{1-\tan^2\alpha}{2\tan\alpha} = \frac{1-\frac{A^2}{C^2}}{\frac{2A}{C}} = \frac{\frac{C^2-A^2}{C^2}}{\frac{2A}{C}} = \frac{C^2-A^2}{2AC} = (19). \quad \text{Q. E. D.}$$

Substituting in (17) values of y_1 and z_1 taken from (2), and making use of (5), (8), and (9), we have

$$\tan^2\varrho\sin^2\varphi = \tan^2\beta\sin^2\alpha - \frac{\tan^2\beta}{\tan^2\alpha}\cos^2\alpha + \sin 2\alpha\left(\tan^2\beta+\frac{\tan^2\beta}{\tan^2\alpha}\right)\tan\varrho\cos\varphi,$$

$$\frac{\tan^2\varrho\sin^2\varphi}{\tan^2\beta} = \frac{\sin^4\alpha-\cos^4\alpha}{\sin^2\alpha} + 2\cos\alpha\sin\alpha\operatorname{cosec}^2\alpha\tan\varrho\cos\varphi,$$

$$\frac{\tan^2\varrho\sin^2\varphi}{\tan^2\beta} = 1 - \cot^2\alpha + 2\cot\alpha\tan\varrho\cos\varphi, \qquad (20)$$

which is the equation of a spherical parabola, α being on the axis of the curve, and the distance of the vertex from the origin being $\frac{1}{2}\pi - 2\alpha$.

9. *Discussion of* (20).— From (20) we get, by solving the quadratic equation,

$$\tan\rho = \cot\alpha \tan^2\beta \frac{\cot\varphi}{\sin\varphi} \pm \sqrt{\left(\frac{\tan^2\beta(1-\cot^2\alpha) + \cot^2\alpha \tan^4\beta \cot^2\varphi}{\sin^2\varphi}\right)}; \quad (21)$$

so that here, as in the other curves, we get four values of ρ for every value of φ for which $\tan\rho$ is real. When $\varphi = 0$, (20) gives

$$\tan\rho = \frac{\cot^2\alpha - 1}{2\cot\alpha} = \frac{\tan^2(\frac{1}{2}\pi - \alpha) - 1}{2\tan(\frac{1}{2}\pi - \alpha)} = -\operatorname{cotan}(\pi - 2\alpha)$$
$$= \tan(\tfrac{1}{2}\pi - 2\alpha), \quad \rho = \tfrac{1}{2}\pi - 2\alpha, \quad \text{or,} \quad = \tfrac{3}{2}\pi - 2\alpha,$$

which gives the points at the vertices of the two curves; for evidently this equation is the equation of the two intersections on the sphere, thus requiring two plane parabolas to make one spherical parabola.

The limits for the value of φ are determined by the equation $1 - \cot^2\alpha + \cot^2\alpha \tan^2\beta \cot^2\varphi = 0$, taken from the radical in (21). This equation gives

$$\cot^2\varphi = \frac{\cot^2\alpha - 1}{\cot^2\alpha \tan^2\beta} = \frac{1 - \tan^2\alpha}{\tan^2\beta} = \cot^2\beta - \cot^2\beta_1$$
$$= \frac{C^2}{B^2} - \frac{A^2}{B^2} = \frac{C^2 - A^2}{B^2}, \qquad \text{or,} \qquad \tan\varphi = \pm\frac{B}{\sqrt{(C^2 - A^2)}}.$$

Hence, $\varphi = \tan^{[-1]}\left[\pm\frac{B}{\sqrt{(C^2-A^2)}}\right]$, or, $\pm\left(\pi + \tan^{[-1]}\left[\frac{B}{\sqrt{(C^2-A^2)}}\right]\right)$.

So that ρ is imaginary for all values of φ between

$$\tan^{[-1]}\left(\frac{B}{\sqrt{(C^2-A^2)}}\right) \quad \text{and} \quad \pi - \tan^{[-1]}\left(\frac{B}{\sqrt{(C^2-A^2)}}\right),$$

and between

$$\tan^{[-1]}\left(-\frac{B}{\sqrt{(C^2-A^2)}}\right) \quad \text{and} \quad \pi + \tan^{[-1]}\left(\frac{B}{\sqrt{(C^2-A^2)}}\right),$$

and real for all others.

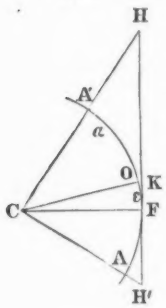
Since here, as in all the other curves, the absolute value of ρ is the same whether φ is positive or negative, the two branches of the curve must be symmetrical.

10. From these investigations, it is evident that any conic section, made by a plane tangent to the sphere whose centre is at the vertex, may be projected on the sphere in the shape of a spherical conic. Moreover, although the curves themselves remain the same, the peculiar characteristics which they assume as a spherical conic — i. e. whether an ellipse, parabola, or hyperbola — are determined solely by the position of the point from which we start as the origin, and at which the cutting plane is tangent.

It must be observed, that, though the ellipse and parabola, when projected on the sphere, give in each case a complete spherical oval, this is not true of each branch of the hyperbola. Each branch, when projected on its own side alone, gives only a portion of the curve, but these portions are complements of each other. But since the tan ρ for any point is the same as for the point diametrically opposite, each curve is twice projected, and thus we get both the curves actually made by the intersections on the sphere.

11. Both the spherical curves we have considered are of double curvature, except when in (4) $A = B = C$, in which case the cone is a right cone, the ellipses are circles, both the hyperbolas are equilateral, and the origin of the parabola, O'', is midway between O' and O.

Fig. 4.



12. *Equation of the Spherical Ellipse referred to the Focus.* — Let us now suppose that the cutting plane of the ellipse is so turned that the point of contact with the sphere is one of the foci. Then F (Fig. 4) is the focus, and $AO = A'O = \alpha$. Denote the arc FO by ε, and let HH' be the greater axis of the ellipse. Then

$$FH = \tan(\alpha + \varepsilon), \qquad FH' = \tan(\alpha - \varepsilon),$$

and $$HH' = 2a = \tan(\alpha + \varepsilon) + \tan(\alpha - \varepsilon)$$

$$= \frac{\sin(\alpha+\varepsilon)\cos(\alpha-\varepsilon) + \sin(\alpha-\varepsilon)\cos(\alpha+\varepsilon)}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)} = \frac{2\sin\alpha\cos\alpha}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)};$$

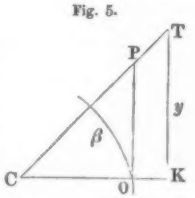
$$\therefore\ a = \frac{\sin\alpha\cos\alpha}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}. \qquad (22)$$

If we represent by m and n the segments of the transverse axis made by any ordinate y, in a plane ellipse, we have, from the well-known properties of the curve, $\frac{y^2}{mn} = \frac{b^2}{a^2}$. Let $m = KH'$ (Fig. 4) and $n = KH$; then

$$\left.\begin{aligned} m &= \tan\varepsilon + \tan(\alpha - \varepsilon) = \frac{\sin\alpha}{\cos\varepsilon\cos(\alpha-\varepsilon)}, \\ \text{and}\qquad n &= \tan(\alpha+\varepsilon) - \tan\varepsilon = \frac{\sin\alpha}{\cos\varepsilon\cos(\alpha+\varepsilon)}. \end{aligned}\right\} \qquad (23)$$

Now, since y is perpendicular to CO, it must be in the same plane with the semi-conjugate axis β, and we have from the similar triangles in Fig. 5 the proportion $CO : CK = OP : KT$. But $CO = 1$, $CK = \sec\varepsilon$, $OP = \tan\beta$, and $KT = y$, and we have $y = \tan\beta\sec\varepsilon$. Therefore, by (23),

Fig. 5.



$$\frac{y^2}{mn} = \frac{\tan^2\beta\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}{\sin^2\alpha} = \frac{b^2}{a^2}.$$

But by (22)

$$\frac{b^2}{a^2} = \frac{b^2\cos^2(\alpha+\varepsilon)\cos^2(\alpha-\varepsilon)}{\sin^2\alpha\cos^2\alpha} = \frac{\tan^2\beta\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}{\sin^2\alpha};$$

hence, $$b^2 = \frac{\tan^2\beta\cos^2\alpha}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}. \qquad (24)$$

The polar equation of the plane ellipse, the origin being at one focus, and the axis directed from the other, is $r = \frac{b^2}{a + c\cos\varphi}$; hence

$$\frac{1}{r} = \cot\varrho = \frac{a}{b^2} + \frac{c}{b^2}\cos\varphi. \qquad (25)$$

From (Fig. 4), $c = \frac{1}{2}(HF - H'F) = \frac{1}{2}(\tan(\alpha+\varepsilon) - \tan(\alpha-\varepsilon))$

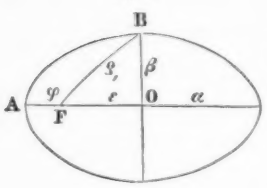
$$= \frac{\sin\varepsilon\cos\varepsilon}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}. \qquad (26)$$

The substitution of (22), (24), and (26) in (25) gives

$$\cot\rho = \frac{\sin\alpha\cos\alpha + \sin\varepsilon\cos\varepsilon\cos\varphi}{\tan^2\beta\cos^2\alpha}, \qquad (27)$$

which is the equation of a spherical ellipse, the origin being at one focus, and the polar axis directed from the other.

Fig. 6.



13. *To prove that* ρ_1, *the radius vector drawn to the extremity of the conjugate axis, equals* α. — Let Fig. 6 represent the spherical ellipse, and we have, by NAPIER'S Rule,

$$\cos\varphi = -\tan\varepsilon\cot\rho_1,$$

which, substituted in (27), gives

$$\cot\rho_1 = \frac{\sin\alpha\cos\alpha - \sin^2\varepsilon\cot\rho_1}{\tan^2\beta\cos^2\alpha} = \frac{\cos\alpha\sin\alpha}{\tan^2\beta\cos^2\alpha + \sin^2\varepsilon}. \qquad (28)$$

Now when $\varphi = 0$, $\rho = \alpha - \varepsilon$, and (27) becomes

$$\cot(\alpha-\varepsilon) = \frac{\sin\alpha\cos\alpha + \cos\varepsilon\sin\varepsilon}{\tan^2\beta\cos^2\alpha} = \frac{\sin 2\alpha + \sin 2\varepsilon}{2\tan^2\beta\cos^2\alpha}$$

$$= \frac{\sin(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}{\tan^2\beta\cos^2\alpha} = \frac{\cos(\alpha-\varepsilon)}{\sin(\alpha-\varepsilon)}; \qquad (29)$$

hence, $\sin(\alpha+\varepsilon)\sin(\alpha-\varepsilon) = \tan^2\beta\cos^2\alpha.$

By trigonometry, $\sin(\alpha+\varepsilon)\sin(\alpha-\varepsilon) = \frac{1}{2}(\cos(2\varepsilon) - \cos 2\alpha)$;

and since $\cos 2\alpha = 2\cos^2\alpha - 1$, and $\cos 2\varepsilon = 2\cos^2\varepsilon - 1$,

we have $\cos^2\varepsilon - \cos^2\alpha = \tan^2\beta\cos^2\alpha,$

or, $\cos^2\varepsilon = \cos^2\alpha + \tan^2\beta\cos^2\alpha;$ (30)

hence, $\sin^2\varepsilon = 1 - \cos^2\varepsilon = 1 - \cos^2\alpha - \tan^2\beta\cos^2\alpha,$

which, substituted in (28), gives

$$\cot \rho_1 = \frac{\cos \alpha \sin \alpha}{\sin^2 \alpha} = \cot \alpha; \qquad \therefore \rho_1 = \alpha. \quad \text{Q. E. D.}$$

14. *To prove that the sum of two radii vectores, drawn from the two foci to a common point in the curve, is equal to* 2α. — Let ρ and ρ' be the two radii drawn from the two foci to the common point; and since $FF' = 2\varepsilon$, we have, by trigonometry,

$$\cos \rho' = \cos 2\varepsilon \cos \rho - \sin \rho \sin 2\varepsilon \cos \varphi. \qquad (31)$$

From (30) we get $\cos^2 \varepsilon = \sec^2 \beta \cos^2 \alpha$, which, multiplied by $\sin \beta$, gives $\sin^2 \beta \cos^2 \varepsilon = \tan^2 \beta \cos^2 \alpha$; and, by substitution, (27) gives

$$\cos \varphi = \frac{\sin^2 \beta \cos^2 \varepsilon \cot \rho - \sin \alpha \cos \alpha}{\sin \varepsilon \cos \varepsilon};$$

and by this equation (31) becomes

$$\begin{aligned}
\cos \rho' &= \cos 2\varepsilon \cos \rho - \frac{\sin^2 \beta \cos^2 \varepsilon \cos \rho \sin 2\varepsilon - \sin \rho \sin 2\varepsilon \sin \alpha \cos \alpha}{\sin \varepsilon \cos \varepsilon} \\
&= \cos 2\varepsilon \cos \rho - 2 \sin^2 \beta \cos^2 \varepsilon \cos \rho + \sin \rho \sin 2\alpha \\
&= \cos \rho (\cos 2\varepsilon - 2 \sin^2 \beta \cos^2 \varepsilon) + \sin \rho \sin 2\alpha \\
&= \cos \rho (2 \cos^2 \varepsilon - 1 - 2 \sin^2 \beta \cos^2 \varepsilon) + \sin \rho \sin 2\alpha \\
&= \cos \rho (2 \cos^2 \varepsilon \cos^2 \beta - 1) + \sin \rho \sin 2\alpha.
\end{aligned}$$

From (30) we get $\cos^2 \alpha = \cos^2 \varepsilon \cos^2 \beta$, so that the above becomes

$$\begin{aligned}
\cos \rho' &= \cos \rho (2 \cos^2 \alpha - 1) + \sin \rho \sin 2\alpha \\
&= \cos \rho \cos 2\alpha + \sin \rho \sin 2\alpha \\
&= \cos (\rho - 2\alpha) = \cos (2\alpha - \rho);
\end{aligned}$$

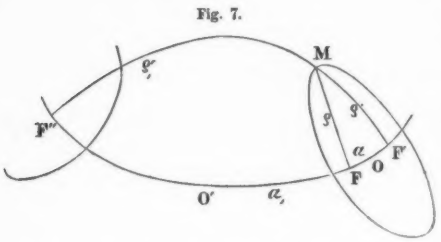
$$\therefore \rho' = 2\alpha - \rho, \qquad \text{or,} \qquad \rho' + \rho = 2\alpha. \quad \text{Q. E. D.}$$

15. It is impossible to obtain directly the equation of the spherical hyperbola, referred to either focus; or the equation of the parabola referred to its focus, as has been done in the case of the ellipse, for

in either case any plane tangent to the sphere at a focus cuts the cone only in an ellipse. We can, however, show that some of the properties of the curves already shown to form the two branches of a spherical hyperbola are analogous to those of the plane hyperbola.

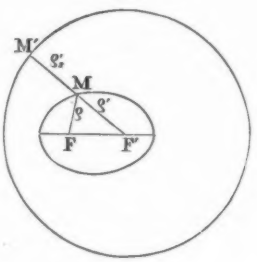
Let F and F' (Fig. 7) be the foci of one spherical ellipse, and F'' a focus in the other, distant from F' by the arc $F''FF' = F''MF' = \pi$.

Fig. 7.



Then F and F'' are the foci of the spherical hyperbola, and ϱ_1' and ϱ are the radii vectores, drawn from the foci to a common point M. It is required to prove that $\varrho_1' - \varrho = 2\alpha_1$. We know that $\varrho_1' + \varrho' = \pi$, and by the preceding section $\varrho + \varrho' = 2\alpha$. Hence, $\varrho' = 2\alpha - \varrho$, and $2\alpha - \varrho + \varrho_1' = \pi$, or $\varrho_1' - \varrho = \pi - 2\alpha = 2\alpha_1$. Moreover, if from one of the foci of a spherical ellipse as a centre, with a radius 2α, a circle be described, it will be the directrix of the curve considered as a spherical parabola, with the other focus as the focus of the parabola. For, since (Fig. 8)

Fig. 8.



$$F'M' = F'M + MM' = 2\alpha$$
$$= F'M + MF = \varrho + \varrho',$$

we have $$FM = MM',$$

or, $$\varrho = \varrho_2';$$

i. e. every point of the curve is equally distant from the focus and the directrix.

EQUATION AND CONSTRUCTION OF THE HYPERBOLA.

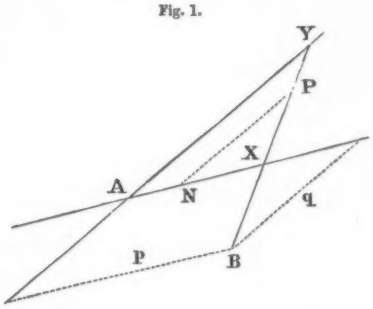
By JOHN WARNER, Philadelphia, Pa.

Two straight lines, AX, AY, and a point B, are given in position. Required the locus of the point P, which divides in a given ratio the intercept XY, included between the given straight lines upon any straight line drawn through B.

Take AX and AY (Fig. 1) for the axes, A for the origin at the intersection of the given straight lines; and let p and q be the co-ordinates of the point B. Let XY be always so divided that

$$XY = m\,XP.$$

Fig. 1.



Let the co-ordinates of P be $x = AN$, $y = PN$; the co-ordinates of X be $x' = AX$, $y' = 0$; and the co-ordinates of Y, $x'' = 0$, $y'' = AY$. The similar triangles, YAX, PNX, give $AX : NX :: XY : XP :: m : 1$; or, $x' : NX :: m : 1$; whence $NX = \frac{x'}{m}$, therefore $x = AN = AX - NX = x' - \frac{x'}{m}$, and $x' = \frac{m}{m-1}x$. Also, from the same similar triangles,

$$AY : PN :: XY : XP :: m : 1, \quad \text{or,} \quad y'' : y :: m : 1,$$

and $y'' = my$. Hence the preliminary equations

$$y'' = my, \qquad \text{and} \qquad x' = \frac{m}{m-1}\,x. \tag{1}$$

The equation of BY which passes through B and Y, of which the co-ordinates respectively are p, q, and x'', y'', is

$$y'' - y = \frac{y'' - q}{p - x''}\,(x - x'').$$

Wherein $x'' = 0$ gives

$$y'' - y = \frac{y'' - q}{p}\, x. \tag{2}$$

And because $B\,Y$ passes through X and Y, of which the co-ordinates respectively are x', y', and x'', y'', its equation is

$$y'' - y = \frac{y'' - y'}{x'' - x'}(x'' - x).$$

Wherein $y' = 0$, $x'' = 0$, gives

$$y'' - y = \frac{y''}{x'}\, x. \tag{3}$$

By equating (2) and (3), we find $\frac{y''}{x'} = \frac{y'' - q}{p}$; whence,

$$x'\,y'' - p\,y'' - q\,x' = 0. \tag{4}$$

But we have, by hypothesis and by (1), $y'' = m\,y$, and $x' = \frac{m}{m-1}\,x$; hence (4) becomes, by substitution,

$$\frac{m}{m-1}\,x\,y - p\,y - \frac{q}{m-1}\,x = 0, \tag{5}$$

which is the equation of the locus of P. If the origin be transposed to a point of which the co-ordinates are $x''' = \frac{m-1}{m}\,p$, $y''' = \frac{q}{m}$, the resulting equation will refer the curve to axes parallel to the original axes, and equation (5) will become

$$x\,y = (m-1)\,\frac{p\,q}{m^2}, \tag{6}$$

which is the equation of an hyperbola referred to the centre and to the asymptotes.

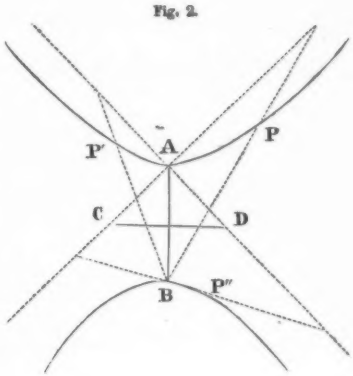
By making m variable, other curves may be generated. The foregoing investigation suggests the following construction.

To construct an hyperbola, the axes being given.

Let $A\,B$ be the transverse axis, and $C\,D$ the conjugate. Join one

end of the transverse, as A, with C and D, and prolong the lines AC, AD, each in both directions. From the other end of the transverse draw BP, BP', BP'', to intersect AC and AD; and bisect the intercepts included between these two lines upon the radii BP, BP', BP''. The points P, P', P'' will belong to an hyperbola, as above shown. The asymptotes will be parallel to AC and AD; but it is known that AC and AD are parallel to the asymptotes of that hyperbola of which the axes are AB and CD.

Fig. 2.



In this construction we have, evidently, $p = q = AD = AC$, and m has been taken $= 2$; hence equation (6) becomes

$$xy = \tfrac{1}{4}\overline{AC}^2 = \frac{a^2+b^2}{4},$$

which is a known property of the hyperbola, a and b being put for the semi-axes.

CONVENIENT FORMULÆ FOR INTERPOLATION.

By William Ferrel, Nautical Almanac Office, Cambridge, Mass.

Let δ^1, δ^2, δ^3, ... δ^s be the first, second, &c. orders of differences of the numbers given for interpolation, and N_0, N_1, N_2 ... N_n the numbers after interpolation, N_0 and N_m corresponding with the numbers between which the interpolations are made. Also let Δ_1, Δ_2, Δ_3 ... Δ_n be the first differences of the interpolated numbers, the

interpolations being supposed to be made at equal intervals. We have, then, the well-known equation

$$N_n = \frac{n}{m}\,\delta^1 + \frac{n}{m}\cdot\frac{n-m}{2m}\,\delta^2 + \frac{n}{m}\cdot\frac{n-m}{2m}\cdot\frac{n-2m}{3m}\,\delta^3 + \&c. \qquad (1)$$

Put $P_n^{(e)}$ for the coefficient of δ^e in the value of N_n, then

$$\left.\begin{aligned} P_n^{(e)} &= \frac{n}{m}\cdot\frac{n-m}{2m}\cdot\frac{n-2m}{3m}\cdots\frac{n-(e-1)m}{em}, \\ P_{n-1}^{(e)} &= \frac{n-1}{m}\cdot\frac{n-1-m}{2m}\cdot\frac{n-1-2m}{3m}\cdots\frac{n-1-(e-1)m}{em}. \end{aligned}\right\} \qquad (2)$$

Put
$$Q_n^{(e)} = P_n^{(e)} - P_{n-1}^{(e)}; \qquad (3)$$

then
$$Q_n^{(e)} = Q_n^{(e-1)}\left(\frac{n-1-(e-1)m}{em}\right) + \frac{P_n^{(e-1)}}{em}, \qquad (4)$$

and
$$\Delta_n = N_n - N_{n-1} = \Sigma_e\, Q_n^e\, \delta^e. \qquad (5)$$

From equation (4) we obtain, by substituting for e the values 1, 2, 3, &c. successively, and transforming the results,

$$\left.\begin{aligned} Q_n^{(1)} &= \frac{1}{m}, \\ Q_n^{(2)} &= -\beta\cdot\frac{1}{2m^2}. \\ Q_n^{(3)} &= \quad \beta\cdot\frac{1}{4m^2} + \gamma\,\frac{1}{6m^3}, \\ Q_n^{(4)} &= -\beta\cdot\frac{4m^2-m+1}{24m^4} - \gamma\cdot\frac{1}{6m^3} + E_n^{(4)}, \\ Q_n^{(5)} &= \quad \beta\cdot\frac{2m^2-m+1}{16m^4} + \gamma\cdot\frac{1}{6m^3} + E_n^{(5)}, \end{aligned}\right\} \qquad (6)$$

in which

$$\left.\begin{aligned} \beta &= (m+1) - 2n, \\ \gamma &= \tfrac{1}{2}m^2 + \tfrac{3}{2}m + 1 + 3n^2 - 3n - 3mn, \end{aligned}\right\} \qquad (7)$$

$$E_n^{(4)} = \frac{2n^3 - 3n^2 + n - 3mn^2 + 4mn + m^2n - m^2 - m}{12m^4},$$

$$E_n^{(5)} = \frac{10n^4 - 20n^3 + 20n^2 - 10n - 50mn + 120mn^2 - 120n^2n - 20m^3n - 80mn^3 - 90m^2n^2 + 5m + 30m^2 + 25m^3 - m^4}{240\,m^5}.$$

With these values, equation (5) gives, carrying the summation to the fifth order of differences,

$$\Delta_n = A + \beta B + \gamma C + E_n^{(4)} + E_n^{(5)}, \tag{8}$$

in which

$$\left.\begin{aligned} A &= \frac{1}{m}\delta^1, \\ B &= -\frac{1}{2m^2}\delta^2 + \frac{1}{4m^2}\delta^3 - \frac{4m^2-m+1}{24m^4}\delta^4 + \frac{2m^2-m+1}{16m^4}\delta^5, \\ C &= \frac{1}{6m^3}(\delta^3 - \delta^4 + \delta^5). \end{aligned}\right\} \tag{9}$$

The terms $E_n^{(4)}$ and $E_n^{(5)}$ are very small, and in all usual applications of the formula, in which only five orders of differences are taken in, are entirely insensible.

By substituting successively 1, 2, 3, &c. for n in the functions β and γ in the general equation (8), and neglecting the last two terms, the following special formulæ of most usual application are obtained. In most cases, the function γ for all values of n can be divided by an integer number, and C be multiplied by the same, which makes the coefficients of C in the formulæ smaller. In the last case γ is rendered divisible by 6 by deducting unity, and the effect of the same is added to the constant A. When m is an odd number, β can also be divided by 2, in which case the value of B must be doubled. In the following formulæ, small changes in some cases have been made in the fractional coefficients of δ^4 and δ^5, at a very slight expense of accuracy, in order to obtain a fraction more convenient in computation, and the maximum errors in any of the interpolated numbers arising from these changes, and the neglect of the terms $E^{(4)}$ and $E^{(5)}$, are denoted by e for those of the fourth differences, and by e' for those of the fifth differences. The values of Δ_1, Δ_2, Δ_3, &c. being obtained by the formulæ, the interpolated numbers are easily written out. A great advantage in these formulæ is, that the number of

terms used in interpolating, arising from all the orders of differences used, is much diminished; and of these, one is constant in all the values of Δ, and the others are very simple multiples of the fundamental quantities B and C, and the one half is obtained from the other half by simply changing the signs, or reversing their order.

When $m = 3$,

$$\Delta_1 = A + 1B + 1C, \qquad \Delta_3 = A - 1B + 1C;$$
$$\Delta_2 = A - 0B - 2C,$$

in which
$$A = \tfrac{1}{3}\delta^1,$$
$$B = -\tfrac{1}{9}\delta^2 + \tfrac{1}{18}\delta^3 - \tfrac{1}{28}\delta^4 + \tfrac{1}{40}\delta^5,$$
$$C = \tfrac{1}{162}(\delta^3 - \delta^4 + \delta^5),$$
$$e = \tfrac{1}{1350}\delta^4, \qquad e' = \tfrac{1}{1000}\delta^5.$$

When $m = 4$,

$$\Delta_1 = A + 3B + 1C, \qquad \Delta_3 = A - 1B - 1C,$$
$$\Delta_2 = A + 1B - 1C, \qquad \Delta_4 = A - 3B + 1C;$$

in which
$$A = \tfrac{1}{4}\delta^1,$$
$$B = -\tfrac{1}{32}\delta^2 + \tfrac{1}{64}\delta^3 - \tfrac{1}{100}\delta^4 + \tfrac{1}{140}\delta^5,$$
$$C = \tfrac{1}{128}(\delta^3 - \delta^4 + \delta^5),$$
$$e = \tfrac{1}{1075}\delta^4, \qquad e' = \tfrac{1}{808}\delta^5.$$

When $m = 5$,

$$\Delta_1 = A + 2B + 2C, \qquad \Delta_4 = A - 1B - 1C,$$
$$\Delta_2 = A + 1B - 1C, \qquad \Delta_5 = A - 2B + 2C;$$
$$\Delta_3 = A \quad 0B - 2C,$$

in which
$$A = \tfrac{1}{5}\delta^1,$$
$$B = -\tfrac{1}{26}\delta^2 + \tfrac{1}{50}\delta^3 - \tfrac{1}{80}\delta^4 + \tfrac{1}{110}\delta^5,$$
$$C = \tfrac{1}{250}(\delta^3 - \delta^4 + \delta^5),$$
$$e = \tfrac{1}{1670}\delta^4, \qquad e' = \tfrac{1}{765}\delta^5.$$

When $m = 6$,

$$\begin{aligned} \Delta_1 &= A + 5B + 5C, & \Delta_4 &= A - 1B - 4C, \\ \Delta_2 &= A + 3B - 1C, & \Delta_5 &= A - 3B - 1C, \\ \Delta_3 &= A + 1B - 4C, & \Delta_6 &= A - 5B + 5C; \end{aligned}$$

in which
$$\begin{aligned} A &= \tfrac{1}{6}\,\delta^1, \\ B &= -\tfrac{1}{72}\,\delta^2 + \tfrac{1}{144}\,\delta^3 - \tfrac{1}{2160}\,\delta^4 + \tfrac{1}{8640}\,\delta^5, \\ C &= \tfrac{1}{6480}\,(\delta^3 - \delta^4 + \delta^5), \\ e &= \tfrac{1}{10800}\,\delta^4, \qquad e' = \tfrac{1}{64800}\,\delta^5. \end{aligned}$$

When $m = 8$,

$$\begin{aligned} \Delta_1 &= A + 7B + 7C, & \Delta_5 &= A - 1B - 5C, \\ \Delta_2 &= A + 5B + 1C, & \Delta_6 &= A - 3B - 3C, \\ \Delta_3 &= A + 3B - 3C, & \Delta_7 &= A - 5B + 1C, \\ \Delta_4 &= A + 1B - 5C, & \Delta_8 &= A - 7B + 7C; \end{aligned}$$

in which
$$\begin{aligned} A &= \tfrac{1}{8}\,\delta^1, \\ B &= -\tfrac{1}{128}\,\delta^2 + \tfrac{1}{256}\,\delta^3 - \tfrac{1}{4000}\,\delta^4 + \tfrac{1}{48000}\,\delta^5, \\ C &= \tfrac{1}{10240}\,(\delta^3 - \delta^4 + \delta^5), \\ e &= \tfrac{1}{12288}\,\delta^4, \qquad e' = \tfrac{1}{48000}\,\delta^5. \end{aligned}$$

When $m = 10$,

$$\begin{aligned} \Delta_1 &= A + 9B + 6C, & \Delta_6 &= A - 1B - 4C, \\ \Delta_2 &= A + 7B + 2C, & \Delta_7 &= A - 3B - 3C, \\ \Delta_3 &= A + 5B - 1C, & \Delta_8 &= A - 5B - 1C, \\ \Delta_4 &= A + 3B - 3C, & \Delta_9 &= A - 7B + 2C, \\ \Delta_5 &= A + 1B - 4C, & \Delta_{10} &= A - 9B + 6C; \end{aligned}$$

in which
$$\begin{aligned} A &= \tfrac{1}{8}\,\delta^1, \\ B &= -\tfrac{1}{2000}\,\delta^2 + \tfrac{1}{4000}\,\delta^3 - \tfrac{1}{61440}\,\delta^4 + \tfrac{1}{86400}\,\delta^5, \\ C &= \tfrac{1}{100000}\,(\delta^3 - \delta^4 + \delta^5), \\ e &= \tfrac{1}{60000}\,\delta^4, \qquad e' = \tfrac{1}{300000}\,\delta^5, \end{aligned}$$

When $m = 12$,

$$\begin{aligned}
\Delta_1 &= A + 11B + 9C, & \Delta_7 &= A - 1B - 6C,\\
\Delta_2 &= A + 9B + 4C, & \Delta_8 &= A - 3B - 5C,\\
\Delta_3 &= A + 7B \quad 0C, & \Delta_9 &= A - 5B - 3C,\\
\Delta_4 &= A + 5B - 3C, & \Delta_{10} &= A - 7B \quad 0C,\\
\Delta_5 &= A + 3B - 5C, & \Delta_{11} &= A - 9B + 4C,\\
\Delta_6 &= A + 1B - 6C, & \Delta_{12} &= A - 11B + 9C;
\end{aligned}$$

in which
$$A = \tfrac{1}{12}\delta^1 + \tfrac{1}{6}C,$$
$$B = -\tfrac{1}{288}\delta^2 + \tfrac{1}{576}\delta^3 - \tfrac{1}{8800}\delta^4 + \tfrac{1}{12000}\delta^5,$$
$$C = \tfrac{1}{1728}(\delta^3 - \delta^4 + \delta^5),$$
$$e = \tfrac{1}{5500}\delta^4, \qquad e' = \tfrac{1}{3000}\delta^5.$$

The following is an example of the application of the last formula to the interpolation of the moon's right-ascension from half-days to hours. It is taken where the fourth and fifth differences are unusually large.

R. A.	δ^1	δ^2	δ^3	δ^4	δ^5	A.	B.	C.
h m s	m s	s	s	s	s	m s	s	s
2 12 47.60	+29 24.67	+19.59	−5.17	−1.81	+0.52	+2 27.056	−0.0753	−0.0016
2 42 12.27	29 44.26	14.42	6.98	1.29	0.69	2 28.688	0.0602	0.0028
3 11 56.53	29 58.68	7.44	8.27	0.60	0.80	2 29.890	0.0386	0.0036
. . . . .	. . . . .	. . .	. . .	. . .	. . .	. . . . .	. . . . .	. . . . .

In the first of the following columns are written the functions $11B$, $9B$, &c., and in the second, $9C$, $4C$, &c. The sum of these is placed in the third column, to which the constant A is added for Δ^1, or the fourth column, the numbers of which are the first differences of the interpolated numbers. The differences of Δ^1, i. e. Δ^2, should be taken as a test of the accuracy of the work before writing out the interpolated numbers.

Great care should be taken to get the quantities A, B, and C very accurately, and when a series of interpolations are to be made, they should be tested by means of the differences, before proceeding to

interpolate, in order to save the trouble of correcting whole groups of numbers, when the errors are discovered by the subsequent tests.

Functions of B.	Functions of C.	Sum of Functions.	Δ^1	Δ^2	R. A.
			m s		h m s
—.828	—.014	—.842	2 26.214	.158	2 12 47.60
.678	.006	.684	.372	.157	15 13.81
.527	—.000	.527	.529	.156	17 40.18
.376	+.005	.371	.685	.153	20 6.71
.226	.008	.218	.838	.153	22 33.39
—.075	.010	—.065	.991	.150	25 0.23
+.075	.010	+.085	27.141	.149	27 27.22
.226	.008	.234	.290	.147	29 54.36
.376	.005	.381	.437	.146	32 21.65
.527	+.000	.527	.583	.145	34 49.09
.678	—.006	.672	.728	.142	37 16.67
+.828	.014	+.814	.870	.131	39 44.40
—.662	.025	—.687	28.001	.134	42 12.27
.542	.011	.553	.135	.132	44 40.27
.421	—.000	.421	.267	.128	47 8.40
.301	+.008	.293	.395	.126	49 36.67
.181	.014	.167	.521	.124	52 5.07
—.060	.017	—.043	.645	.120	54 33.59
+.060	.017	+.077	.745	.118	57 2.23
.181	.014	.195	.883	.114	2 59 31.00
.301	.008	.309	.997	.112	3 1 59.88
.421	+.000	.421	29.109	.110	4 28.88
.542	—.011	.531	.219	.106	6 57.99
+.662	—.025	+.637	.325		9 27.21
...	...	...	...	...	3 11 56.53
					. .. .

The maximum error, in this example of the moon's right ascension, from neglecting the terms E^4 and E^5, is only about $0^s.003$, and is generally less than $0^s.002$.

If in equation (8) we substitute for β, γ, A, B, and C, their values in equations (7) and (9), and suppose m to be infinitely great,

Δ_n becomes dN, and $\frac{1}{m}$ becomes $d\frac{n}{m}$, and neglecting terms infinitely small, it becomes

$$m D_n N = \tfrac{1}{2} C' \frac{n^2}{m^2} - (2B' + \tfrac{1}{2} C') \frac{n}{m} + A' + B' + \tfrac{1}{12} C, \quad (10)$$

in which

$$\left.\begin{aligned} A' &= \delta', \\ B' &= -\tfrac{1}{2}\delta^2 + \tfrac{1}{4}\delta^3 - \tfrac{1}{6}\delta^4 + \tfrac{1}{8}\delta^5, \\ C' &= \delta^3 - \delta^4 + \delta^5. \end{aligned}\right\} \quad (11)$$

Putting $D_n N = 0$, we get

$$\frac{n^2}{m^2} - \left(\frac{4B'}{C'} + 1\right)\frac{n}{m} = -2\,\frac{(A' + B')}{C'} - \tfrac{1}{6}. \quad (12)$$

The value of $\frac{n}{m}$ obtained from this equation is the fraction of the interval at which N becomes a maximum or a minimum. By means of this equation the exact place of the moon's perigee or apogee, or of a planet's perihelion or aphelion, the moon's greatest libration, &c., can be very conveniently obtained without interpolating to short intervals.

Integrating equation (10), we get

$$N - N' = \tfrac{1}{6} C' \frac{n^3}{m^3} - (B' + \tfrac{1}{4} C') \frac{n^2}{m^2} + (A' + B' + \tfrac{1}{12} C') \frac{n}{m}, \quad (13)$$

N' being the value of N when $n = 0$. By means of this equation, the exact place or fraction of interval can be determined at which N becomes of a given value, and it can be used in determining the phases of the moon, and in all similar cases.

ALPHABETICAL INDEX.

Page

Algebra, TODHUNTER s, Notes from, 52, 53, 54
——— for the Use of Colleges and Schools, by TODHUNTER, Notice of, . . . 62
——— University, by DAVIES, Notice of, 63
Algebraic Curves, A General Method of finding Tangents to, by HOLLIDAY, . 149
Analysis, On the Diophantine, by A. D. WHEELER, ... 325
——— Spherical, by EASTWOOD, . 78
Arithmetic, A Logical Outline of, by BROOKS, ... 182
——— Mean of any Number of Positive Quantities is greater than the Geometric Mean, ... 69
Astronomical Notices, edited by BRÜNNOW, Notice of, ... 64
Astronomy, A Treatise on, by ROBINSON, Notice of, ... 125
Asteroids, On the Secular Variations and Mutual Relations of the Orbits of, by NEWCOMB, Notice of, ... 95
BARDWELL, Prof. F. W., Note on the Horizontal Thrust of Embankments, . 6
BARROW, ISAAC, D. D., The Mathematical Works of, Notice of, ... 286
Binomial Theorem, Demonstration of, . 54
Binomials, Note on Powers of, . . . 69
BORDEN, JOHN, Note on the Dependence of Equations, ... 7
BROOKS, Prof. EDWARD, A Logical Outline of Arithmetic, ... 182
Calculus, Differential and Integral, by CHURCH, Notice of, ... 289

Page

Calculus, Differential and Integral, by DAVIES, Notice of, ... 30
——— by SMYTH, Notice of, . . . 159
Calculus of Variations, A History of the Progress of, during the Nineteenth Century, by TODHUNTER, Notice of, . . . 354
Chart and Diagram for facilitating Great-Circle Sailing, by GODFRAY, Notice of, 94
Chemical Physics, Elements of, by COOKE, Notice of, ... 30
CHURCH, Prof. A. E., Elements of the Differential and Integral Calculus, Notice of, ... 289
Circle, On an Approximate and Graphical Rectification of, by HERSCHEL, . . 153
——— On the Quadrature of, by SMITH, Notice of, ... 291
Circular Arc, Time of Vibration in, by GIBBES, ... 40
COLLINS, MATTHEW, Mathematical Notes, 306
Common Measure, Note on the Greatest, by HUNT, ... 51
Conics, Spherical, by WOODWARD, . 361
COOKE, JOSIAH P., JR., Elements of Chemical Physics, Notice of, ... 30
Cubes of the first n Natural Numbers, Sum of the, ... 53
Curves, On the Geometrical Construction of, by Points, by NEWTON, . . 235, 268
DAVIES, Prof. CHARLES, Elements of Analytical Geometry and Differential and Integral Calculus, Notice of, . . . 30
——— University Algebra, Notice of, 63

Determinants, A Treatise on, by OLIVER, 86
Diary, The Lady's and Gentleman's for 1861, Notice of, ... 159
——— Problems from, ... 190
Differential Calculus, The Claim of LEIBNITZ to the Invention of, by SLOMAN, Notice of, ... 259
DODD, JAMES B., Series of Mathematical Text-Books, Notice of, ... 290
Earth, On the Conformation of the, Prize Essay, by HILL, ... 166
EASTWOOD, GEORGE, On Spherical Analysis, ... 78
Editorial Items, . . 64, 160, 192, 228, 292
ELLIS, THEODORE G., The Theory of the Gyroscope, ... 209
Embankments, Horizontal Thrust of, by BARDWELL, ... 6
Equations, Note on the General Theory of, by EVANS, ... 256
——— Note on Numerical, by VALLAS, 10
——— Numerical, of the Third Degree, Solution of, by PORTER, . . . 198
——— On the Dependence of, by BORDEN, ... 7
——— which determine the Position of a Comet or other Planetary Body, from Three Observations, Discussion of, by HILL, ... 26
Equilibrium, Demonstration of the Law of, in the Lever, by STRONG, . . . 283
EVANS, ASHER B., Solution of Problem V., No. X., Vol. II., ... 151
——— Prof. E. W., Note on the General Theory of Equations, ... 256
FERREL, WILLIAM, Convenient Formulæ for Interpolation, ... 377
Fraction, Continued, To change a Series into, by HALL, ... 262
Geodetic Formulas, by MATTHEWSON, 71
Geometry, Note on, by POWERS, . . 185
——— Analytical, Note on, by STEVENS, 185
——— Analytical, by DAVIES, Notice of, 30
——— Descriptive, General Problems from the Orthographic Projections of, by WARREN, Notice of, . . . 155
Geometry, Descriptive, of One Plane, by WATSON, ... 133
——— Elements of, by GREENLEAF, Notice of, ... 32
GIBBES, Prof. LEWIS R., An Easy Mode of Approximating to the Time of Vibration in a Circular Arc, ... 40
GODFRAY, HUGH, Note on the Problem of the Lights, ... 38
Great-Circle Sailing, or a Chart and Diagram for facilitating, by GODFRAY, Notice of, ... 94
GREENLEAF, BENJAMIN, Elements of Geometry, Notice of, ... 32
Gyroscope, Theory of, by ELLIS, . . 209
HALL, A., To Change a Series into a Continued Fraction, ... 262
HENKLE, Prof. W. D., Note on Perfect Squares, ... 66
HERSCHEL, A. S., On an Approximate and Graphical Rectification of the Circle, 152
——— Sir JOHN, A Complete Catalogue of the Writings of, ... 220
HILL, G. W., Discussion of the Equations which determine the Position of a Comet or other Planetary Body, from Three Observations, ... 26
——— Prize Essay, On the Conformation of the Earth, ... 166
HOYT, Prof. D. W., Note on some Properties of Square Numbers, . . . 12
——— Note on Prize Solution of Problem III., p. 25, ... 67
HOLLIDAY, WALLER, A General Method of finding Tangents to Algebraic Curves, 149
HUNT, E., Note on the Greatest Common Measure, ... 51
Hyperbola, Equation and Construction of, by WARNER, ... 375
Inclined Plane, Motion on, by SHERWIN, 206
Insurance, by NEWCOMB, ... 343
Interest for Days, Correction of the Methods for Computing Six Per Cent, . . 67
——— Note on a Question in, by PORTER, 255
Interpolation, Convenient Formulæ for, by FERREL, ... 377

Items, Editorial, . 64, 160, 192, 228, 292
Leibnitz, The Claim of, to the Invention of the Differential Calculus, by Sloman, Notice of, ... 259
Lever, Demonstration of the Law of Equilibrium in, by Strong, . . . 283
Magic Table, by Martin, ... 189
Mathematical Monthly Notices, 30, 61, 90, 125, 155, 190, 259, 286, 354
Mathematical Development, Exposition of the Process of, by Paterson, . . 55, 108
Mathematical Notes, by Collins, . . 306
Mathematical Operations, by Robinson, Notice of, ... 90
Mathematical Works of Isaac Barrow, Notice of, ... 286
Matthewson, Dr. R. C., The Earth considered as a Spheroid of Revolution, . 71
Morgan, H. A., A Collection of Problems and Examples, Notice of, . . . 64
Napier's Rules of Circular Parts, Note on, 99
Natural Philosophy, Introductory Course of, by Peck, Notice of, ... 128
Newcomb, S., Solution of Problem V., No. IX., Vol. II., ... 68
——— Notes on Probabilities, . 119, 343
——— On the Secular Variations and Mutual Relations of the Orbits of the Asteroids, Notice of, ... 95
Newton, Prof. H. A., On the Geometrical Construction of certain Curves by Points, ... 335, 268
Note on a Problem in Robinson's Algebra, by Hoyt, ... 189
Note on Successive Derivatives of tan φ, . 70
Notes and Queries, 51, 66, 98, 149, 182, 254
Notes on Probabilities, by Newcomb, 119, 343
Numerical Equations of the Third Degree, Geometrical Method of Solution of, by Porter, ... 203
Oliver, J. E., A Treatise on Determinants, ... 86
Parker, Prof. W. H., Remarks on the Problem of the Lights, ... 34
Paterson, John, Exposition of the Process of Mathematical Development, 55, 108
Paterson, John, On the Results of Accelerated Velocity, ... 357
Peck, Prof. W. G., Introductory Course of Natural Philosophy, Notice of, . . 128
Pendulum, by Sherwin, ... 312
Perfect Squares, Note on, by Henkle, . 66
Polytechnic School at Paris, by Watson, ... 320
Porter, Prof. J. C., Note on a Question in Interest, ... 254
——— Solution of Numerical Equations of the Third Degree, ... 198
Prize Essay, On Central and Disturbing Forces, by Trowbridge, . . . 245
——— On the Conformation of the Earth, by Hill, ... 166
——— Spherical Conics, by Woodward, 361
——— The Method of Projections, by Wright, ... 293
Prize Essays, Report of Committee on, . 160
Prize Problems for Students, 1, 33, 65, 97, 129, 161, 193, 229, 261
——— Solutions of, 18, 46, 104, 130, 162, 194, 230, 280, 349
Prize Solution of Problem III., p. 25, Note on, by Hoyt, ... 67
Probabilities, Notes on, by Newcomb, 119, 343
Problem of the Lights, Note on, by Godfray, ... 38
——— Remarks on, by Parker, . 34
Problem, Solution of, by Trowbridge, . 187
Problems and Examples, A Collection of, by Morgan, Notice of, . . . 64
Problems and Riders for the Year 1860, Cambridge Senate-House, with Solutions of, by Watson and Routh, Notice of, 61
Projections, The Method of, by Wright, . 293
Quadrature of the Circle, by Smith, Notice of, ... 291
Recurring Decimals, Values of, . . 53
Roberson, J. F., Square Numbers ending in 5, 25, 75, ... 150
Robinson, H. N., A Treatise on Astronomy, Notice of, ... 125
——— Concise Mathematical Operations, Notice of, ... 90

ROBINSON, H. N., Elements of Analytical Geometry and the Differential and Integral Calculus, Notice of, . . . 90
ROUTH, E. J., Cambridge Senate-House Problems and Riders, Notice of, . . 61
School, Polytechnic, at Paris, by WATSON, 320
SCOTT, JOHN, Jr., Solution of Problem, 66
SHERWIN, THOMAS, Motion on an Inclined Plane, ... 206
——— The Pendulum, ... 312
SLOMAN, Dr. H., The Claim of LEIBNITZ to the Invention of the Differential Calculus, Notice of, ... 259
Smithsonian Institution, Annual Report of the Board of Regents, Notice of, . . 190
SMYTH, WILLIAM, Elements of the Differential and Integral Calculus, Notice of, 159
Solution of Problem V., No. IX., Vol. II., by NEWCOMB, ... 68
Spherical Analysis, by EASTWOOD, . 78
Square Numbers, Some Properties of, by HOYT, ... 2
Squares of Numbers ending in 5, 25, 75, by ROBERSON, ... 150
STEVENS, Prof. M. C., Notes on Trigonometry and Analytical Geometry, . . 185
STRONG, Dr. THEODORE, Demonstration of the Law of Equilibrium in the Lever, 283
Successive Derivatives of tan φ, Note on, 70
Sum of the Squares of the first n Natural Numbers, ... 52
Technology, Objects and Plan of an Institute of, Notice of, ... 93
TODHUNTER, I., A History of the Progress of the Calculus of Variations during the Nineteenth Century, Notice of, . . 354
TODHUNTER, I., Algebra for the Use of Colleges and Schools, Notice of, . . 62
——— A Treatise on Plane Co-ordinate Geometry, Notice of, ... 62
TODHUNTER'S Algebra, Notes from, 52, 53, 54
Trigonometry, Plane and Spherical, by TODHUNTER, Notice of, ... 95
TROWBRIDGE, DAVID, Second Prize Essay, On Central and Disturbing Forces, 245
VALLAS, Prof. A, Note on Numerical Equations, ... 10
Velocity, Accelerated, On the Results of, by PATERSON, ... 357
Vibration in a Circular Arc, Time of, by GIBBES, ... 40
WARNER, JOHN, Equation and Construction of the Hyperbola, ... 375
WARREN, S. EDWARD, General Problems from the Orthographic Projections of Descriptive Geometry, Notice of, . . 155
WATSON, Rev. H. W., Cambridge Senate-House Problems and Riders, Notice of, ... 61
WATSON, WILLIAM, Descriptive Geometry of One Plane, ... 135
——— The École des Ponts et Chaussées at Paris, ... 351
——— The Polytechnic School at Paris, 320
WHEELER, Rev. A. D., On the Diophantine Analysis, ... 325
WOODWARD, C. M., Fourth Prize Essay, Spherical Conics, ... 361
WRIGHT, ARTHUR W., Third Prize Essay, The Method of Projections, . . . 293
Writings of SIR JOHN HERSCHEL, A Complete Catalogue of, ... 220

Worcester's Quarto Dictionary The Standard

VERY SIGNIFICANT FACTS.

The following recommendations are from some of the most distinguished American and English scholars. They are but a few from many which have been received, testifying to the superiority of Worcester's Quarto Dictionary. These testimonials are of the highest value, for they have all been given during the present year, and after an examination of this work and of that which is endeavoring to hold the position of a rival. The scholars of America and of England, with scarcely an exception, have decided in favor of Worcester. Not a single scholar, equal in authority to any one mentioned below, can be cited as giving after a comparison of the two works, the preference to Webster's Dictionary. We give the testimony: —

From C. C. FELTON, LL.D., *President of Harvard College.*

Aware of the labor and care which had been devoted to this (the department of scientific terms) as well as to other parts of the work, I felt assured that Worcester's Quarto Dictionary would more nearly meet the public wants than any other hitherto published.

My expectations have been more than fulfilled. I find it not only rich beyond example in its vocabulary, but carefully elaborate in all the details, and thoroughly trustworthy as a guide to the most correct and elegant usage of the language.

From the Rev. JOSEPH BOSWORTH, D.D., F.R.S., *Professor of Anglo-Saxon, Oxford, England.*

It is the most complete and practical, the very best, as well as the cheapest English Dictionary that I know.

From GEORGE P. MARSH, LL.D.

The work of Dr. Worcester is unquestionably much superior to any other general dictionary of the language in every one of these particulars (orthography, pronunciation, definition, fulness of vocabulary, and precision and distinctness of definition).

From Rev. W. WHEWELL, D.D., *Master of Trinity College, England.*

I have repeatedly consulted the Dictionary since it has been in my possession, and have seen reason to think it more complete and exact than its predecessors.

From CHARLES RICHARDSON, LL.D., *the oldest living English Lexicographer, England.*

I sincerely hope you may enjoy from your brethren, both in America and England, that tribute of honor to which you have earned so undoubted a title.

From D. R. GOODWIN, D.D., *President of Trinity College, Hartford.*

It was but a short time since that I was led to commend another dictionary as, on the whole, and with some exceptions, the best and most complete thing of the kind within my knowledge. The commendation was honestly given at the time; but now *it must be withdrawn in favor of yours.* I consider your dictionary, in orthography, pronunciation, and definitions, as superior to any of its predecessors.

From Rev. W. B. SPRAGUE, D.D., *of Albany, N. Y.*

My opinion of Worcester's Quarto Dictionary, after having given it as extended an examination as my circumstances would admit, is, that there is no other dictionary in the language that compares with it for completeness, accuracy, comprehensiveness, and precision, and perhaps I ought to add, that I have arrived at this conclusion rather contrary to a preconceived opinion.

From Rev. HENRY A. BOARDMAN, D D., *of Philadelphia.*

I particularly like it (the Dictionary), 1. Because of its very comprehensive character: 2 Because it adheres to the *settled orthography* of our noble language, — discarding those innovations which, however countenanced by certain publishing-houses, have never to any extent been accepted by the scholars of our country.

From LOUIS AGASSIZ, LL.D.

It is of great importance, when the nomenclature of science is gradually creeping into common use, that an English lexicon should embrace as much of it as is consistent with the language we speak. I am truly surprised and highly delighted to find you have succeeded far beyond my expectations in making the proper selection, and combining with it a remarkable degree of accuracy. More could hardly be given except in a scientific cyclopædia.

The following lines are quoted from Harper's Magazine for September. They serve to show very truthfully the comparative value of recent and old commendations: —

"INJUSTICE. — Our attention has been called to a species of injustice of which publishers are sometimes guilty, in publishing commendations of school-books, without giving the dates when they were written. Especially does this merit reproval when these commendations are old, and when it is known that the writers have subsequently commended other and later publications in the same department. It will readily be seen that this is frequently not only an act of injustice to teachers who have had the courtesy to commend a book, but that it is also a fraud upon the public."

SWAN, BREWER, & TILESTON,

131 Washington Street, Boston.

No. 117 Washington Street, Boston, October, 1860.

CROSBY, NICHOLS, LEE & CO.

HAVE JUST PUBLISHED

THE WORKS OF CHARLES LAMB. With a fine Portrait. In Four Volumes. 12mo. Price, $5.00.

This new edition of Lamb is reprinted from the latest London edition, edited by Talfourd, contains his Life, Letters, and Final Memorials, his Essays of Elia, Rosamund Gray, with other Essays and Poems, being the best and most complete edition issued.

It is elegantly printed on paper slightly tinted, and is in every respect an edition worthy of the author

"This is the most convenient and elegant edition of Lamb's Works which has been published, either in England or in this country, and it leaves nothing to be desired in respect to typographical beauty or any other essential of a good library edition." — *North American Review.*

The Three Great Historics, — Hume, Gibbon, and Macaulay.

Elegant Library Editions, beautifully printed on tinted paper, convenient in size, and withal the best and neatest yet published.

THE HISTORY OF ENGLAND, FROM THE INVASION OF JULIUS CÆSAR TO THE ABDICATION OF JAMES THE SECOND, 1688. By DAVID HUME, Esq. A new Edition; with the Author's last Corrections and Improvements. To which is prefixed a short Account of his Life, written by himself. In 6 volumes. 12mo. With a Portrait. Price, $7.50.

SIR JAMES MACKINTOSH *says of Hume:* "His greatest work, and that which claims most attention, was his 'History of England,' which, notwithstanding great defects, will probably be at last placed at the head of historical compositions. No other narrative seems to unite, in the same degree, the two qualities of being instructive and affecting. No historian approached him in the union of the talent of painting pathetic scenes with that of exhibiting comprehensive views of human affairs. ..... His narrative is flowing and various; in common events, short and clear; in great actions, rapid and animated; in affecting incidents, circumstantial and picturesque. His general observations seem always to be required by his subject; the most profound ideas are clothed in a transparent simplicity."

THE HISTORY OF THE DECLINE AND FALL OF THE ROMAN EMPIRE. By EDWARD GIBBON, Esq. With Notes by the Rev. H. H. MILMAN. A new edition. To which is added a complete Index of the whole Work. In 6 volumes. 12mo. With a Portrait. Price, $7.50.

Of the various editions of GIBBON this of MILMAN is uniformly acknowledged to have no rival.

PROF. SMYTH, *in his Lectures on Modern History,* says: "But the whole is notwithstanding, such an assemblage of merits, so various, so interesting, and so rare, that the History of the Decline and Fall must always be considered as one of the most extraordinary monuments that have appeared of the literary powers of a single mind, and its fame can perish only with the civilization of the world."

THE HISTORY OF ENGLAND FROM THE ACCESSION OF JAMES II. By THOMAS BABINGTON MACAULAY. In 4 volumes. 12mo. Price, $5.00.

"It is as fluent and as much colored as Livy, as close and coherent as Thucydides; with far more real condensation and a larger thoughtfulness than either; and quite free from the affected laconisms and sarcasms and epigrams of Tacitus. I do not know that I ever read anything so good as the first forty pages; so clear, comprehensive, and concise, so pregnant with deep thought, so suggestive of great views, and grand and memorable distinctions." — LORD JEFFREY.

CURIOSITIES OF LITERATURE. By ISAAC DISRAELI. With a View of the Life and Writings of the Author, by his Son. In 4 volumes. 12mo. Price, $5.00.

"It embraces a more copious selection of noteworthy facts in regard to books and authors than can be found in almost any other compilation of the kind, while it doubtless owes much of its popularity with all classes of readers to the agreeable form in which these facts are presented" — *North American Review, April,* 1860.

THE NORTH AMERICAN REVIEW. No. CLXXXIX. — For October, 1860.

CONTENTS. — Article I. Homer and his Heroines. II. Climatology. III. Life and Labors of Thomas Prince. IV. Edmund Waller. V. Lord Shaftesbury. VI. Second Volume of Palfrey's "History of New England." VII. Quarantine and Hygiene. VIII. Rush's Occasional Productions. IX. The English Language in America. X. The Origin of Species. XI. An "Inglorious Milton." XII. Critical Notices. XIII. New Publications.

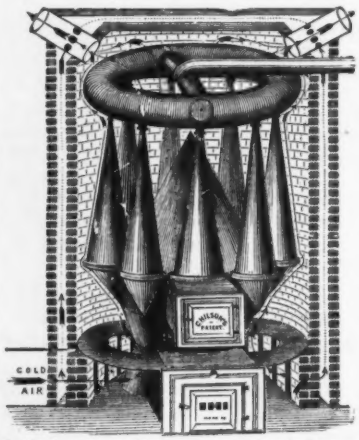
This Review has for nearly fifty years stood at the head of the literature of our country, and has numbered among its contributors most of our eminent literary men. The writings of Webster, Everett, Bancroft, and Prescott were first made known through this work; and its pages are now filled by articles from our most prominent writers.

The NORTH AMERICAN REVIEW is published quarterly, in numbers of nearly three hundred pages each, at five dollars a year.

CHILSON'S
PATENT CONE FURNACE.

PATENTED IN

AMERICA, ENGLAND, AND FRANCE.



THE subscriber begs to call the attention of scientific gentlemen and others to his discovery of a principle for burning fuel, by which all the combustible properties of the smoke and gases eliminated from the burning fuel are thoroughly consumed and their heat retained and made available to the fullest extent, so that there shall be no waste whatever. This important object is accomplished by the use of a large cluster of cones or tapering radiators placed over a broad, shallow, flaring fire-pot, and surmounted by a radiating and exhaustive annular chamber, as represented in the cut. The cones being large at their base and small at the tops, causes the smoke and gases to be suspended directly over and near the fire, where they are thoroughly consumed, and their heat forced to impinge against and up the cones, thereby creating a powerful radiation.

This plan of furnace secures a fresh, healthful heat, free from that scorching, dry, disagreeable, red-hot iron heat and coal-gas so common to hot-air furnaces generally. Persons interested in these matters will here find an invention worthy of their attention, which saves from one half to two thirds of the fuel usually consumed.

COOKING RANGES.

CHILSON'S PATENT DOUBLE OVEN RANGE.

The most important Improvements yet made in Ranges.

THE highly satisfactory testimony received from the large number of these Ranges in use in the past three years should certainly induce those in want to examine this Range, which is *entirely new in principle*, and different from any other before known. It has been got up very perfect and substantial, so that it commends itself at once to the purchaser, in preference to anything else of the Range kind in this country.

Its great economy in the use of fuel, its quick and even baking ovens, its nice arrangement of flues, — without the usual vexatious complication of dampers, — its cast-iron flue plates, preventing any loss of heat in the brick-work, its nice arrangement for roasting meats in the ventilated ovens, &c., will be appreciated by every housekeeper. Sizes in variety for dwellings, hotels, &c., with or without Water Backs for heating water for bathing, &c., Hot Closets, and Hot Air Fixtures for heating additional rooms.

Also, a full assortment of

Chilson's Patent Trio Portable Furnaces, Trio Stoves, Parlor Stoves, Registers, Ventilators, &c., &c.

WAREROOMS 99 & 101 BLACKSTONE ST., BOSTON.
FOUNDERY AT MANSFIELD, MASS.

GARDNER CHILSON.

Philosophical Instruments
AND
SCHOOL APPARATUS,

MANUFACTURED BY

E. S. RITCHIE, BOSTON, MASS.

Few branches of study in our schools are of more practical importance than Natural Philosophy, and it is only by the use of proper apparatus that its principles can be adequately elucidated.

Until a comparatively recent period, the attention bestowed upon this subject has been confined to a few of the elementary principles; of late years, however, the growing interest in physical science, as a branch of common education, has created a desire for more ample and varied means of experimental demonstration.

It has been my earnest desire and effort to keep pace with the progress of natural science, and, as new principles have been discovered or developed, to devise and construct new instruments, to improve the old, and to add to my list many which heretofore could only be procured from Europe; and also that, in superior quality and finish, as well as in graceful form and proportions, whatever goes from my manufactory shall take rank with the best products of European make.

So numerous and important are the changes that have consequently been made, that I have published a new and enlarged **Catalogue of 100 pages octavo,** containing descriptions of over

Twelve Hundred Pieces of Apparatus,

and illustrated by over **Three Hundred** finely executed engravings, and also many commendatory letters from the most distinguished Physicists in the country.

E. S. RITCHIE.

Philosophical Rooms, No. 313 Washington Street, Boston.

☞ Catalogues will be sent, prepaid, by mail to order, — please enclose 25 cents in stamps.

PHOTOGRAPHING ON WOOD.

BRICHER AND RUSSELL,

Photographers, Designers, and Engravers

ON WOOD.

129 WASHINGTON STREET, BOSTON.

BRICHER AND RUSSELL, by their new process of Photographing on Wood, are now enabled to produce much more accurate representations of objects than by the old and slow process of hand drawing. They will furnish Portraits, Views of Colleges, Seminaries, Academies, and Public Buildings generally, College and Society Seals, Philosophical Apparatus, Mathematical Instruments, and Diagrams (see specimens in Mathematical Monthly), Illustrations for School Books of all kinds, &c., &c.

Directions.

Get a Photographer to take a *negative* of the object you wish to have engraved, and of the required size, — that is, send us a photograph of the required size, and the *reverse* of what is wanted when printed.

If a negative photograph cannot be obtained, get as sharp and clear an ambrotype as possible.

The New Englander.

THE NEW ENGLANDER is a QUARTERLY, published in New Haven, and devoted to the discussion of the questions of the day in every department of

THEOLOGY, LITERATURE, AND POLITICS.

In each number there will be full notices of the publications of any importance which have appeared during the preceding three months.

It may not be unadvisable to state that the NEW ENGLANDER has furnished, during the two past years, from two to four hundred pages more than any other three-dollar Quarterly in the country; and no other magazine of the kind attempts to give so full an account of the current literature.

The present is a favorable time to procure COMPLETE SETS of the NEW ENGLANDER from the commencement. These are now becoming very rare. For a short time the few which are on hand will be furnished at rates which can never, probably, be offered again.

Complete Sets of the Nineteen Volumes

will be sold, at **$25**. (Postage or express charge to be at the expense of the purchaser.) Sets complete, *with the exception of six numbers*, at the price of **$16** (without postage or express charge prepaid).

Persons who now have incomplete sets are invited to correspond with the subscriber, who will inform them of the price of the numbers and volumes they need.

The SEVENTEENTH VOLUME (for 1859), 1104 pages, and the EIGHTEENTH VOLUME (for 1860), 1134 pages, are offered at **$2** each (postage prepaid), a price which is below cash cost. In these volumes will be found a fuller account of the new publications of these two years than in any other magazine in the country.

Price of the New Englander for 1861, $3.

When paid in advance, postage will be prepaid in New Haven. Address all communications to

WILLIAM L. KINGSLEY,

Editor and Proprietor,

NEW HAVEN, CONN.

CONTRIBUTIONS

TO THE

Ecclesiastical History

OF

CONNECTICUT.

THE General Association of Connecticut, at the celebration of its One Hundred and Fiftieth Anniversary at Norwich, June, 1859, voted that the Historical Address which was delivered on that occasion should be printed. They also appointed a committee to prepare a volume which should contain, together with the Historical Address, the other addresses which were there delivered, on the Polity, Principles, and Future Prospects of Congregationalism, and all the Historical Papers which were prepared for the occasion.

This volume is now ready for distribution. The title which has been given to it is

"Contributions to the Ecclesiastical History of Connecticut."

It contains an account of all the proceedings at the Celebration at Norwich, June, 1859. The Historical Address, delivered at that time, by Rev. LEONARD BACON, D. D. Twelve Addresses, respecting the Fundamental Principles of Congregationalism, and the Progress and Prospects of the Denomination, by Prof. E. A. LAWRENCE, D. D., East Windsor Theological Seminary; President T. D. WOOLSEY, Yale College; Rev. JOEL HAWES, D. D., Hartford; Rev. T. M. POST, D. D., St. Louis, Missouri; Rev. Prof. E. P. BARROWS, Theological Seminary, Andover, Mass.; Rev. JOHN WADDINGTON, D. D., London, Eng.; Rev. President A. L. CHAPIN, Beloit College, Wisconsin; Rev. S. W. S. DUTTON, D. D., New Haven; Rev. JOSEPH ELDRIDGE, D. D., Norfolk; Rev. SAMUEL WOLCOTT, D. D., Chicago, Ill.; Rev. JOSEPH P. THOMPSON, D. D., New York City; Rev. W. I. BUDINGTON, D. D., Brooklyn, N. Y.

Also twenty-five

"Historical Papers,"

prepared by Rev. MYRON N. MORRIS, West Hartford; Rev. NOAH PORTER, D. D., Farmington; Rev. HORACE HOOKER, Hartford; Rev. CHARLES HYDE, Ellington; Rev. JOEL HAWES, D. D., Hartford; Rev. JOHN MARSH, D. D., New York City; Rev. GEORGE P. PRUDDEN, Watertown; Rev. HIRAM P. ARMS, Norwich Town; Rev. G. A. CALHOUN, D. D., North Coventry; DAVID N. CAMP, Esq.; Rev. R. C. LEARNED, Berlin; Rev. HENRY JONES, Bridgeport; Rev. ABEL MCEWEN, D. D., New London; and others.

In addition, there is a History of each of the fifteen District Associations in the State, with lists of all their licentiates, and a History of each one of the Congregational churches in the State (284 in number), with the names of their Pastors, the dates of their ordination, of their dismission, and death; also a History of the Congregational Churches in the State (21 in number) which have become extinct. No care has been spared to make the volume as complete a history as possible of everything connected with the Congregational denomination in the State of Connecticut.

The volume is a large octavo, contains nearly 600 pages, is bound in cloth, and its price is $3. *It will be sent by mail, or by express, postage prepaid, after the receipt of the money in New Haven by the undersigned.* It is expected that it *will be a constant book of reference* for many years to come.

Address all letters and communications to

WILLIAM L. KINGSLEY,

No. 30 Grove St., New Haven, Conn.

ROSS' AMERICAN SCHOOL FURNITURE WORKS,

Established, 1838.

The Pioneer and most Extensive Establishment in the United States.

The above Engraving represents Ross' Improved High School, Normal School, or Academy Double-Column Desk and Chairs; Falls to lift; plain top.

The above Engraving represents Ross' Improved High School, Normal School, or Academy Single-Column Desk and Chair; Fall to lift; plain top.

The above Engraving represents Ross' Improved High School, Normal School, or Academy Single Desk and Chair; Fall to lift; plain top.

The above Engraving represents Ross' Improved High School, Normal School, or Academy Double Desk and Chairs; Falls to lift; plain top.

An Illustrated Catalogue and Information forwarded, on Application, by Mail or Otherwise.

N. B. *Every Article of School Furniture from this Establishment* WILL BE WARRANTED. *All Communications may be addressed to*

JOSEPH L. ROSS,
Office--CHARDON, opp. HAWKINS STREET,
(*Near the Revere House,*) **BOSTON, Mass.**

Entered according to Act of Congress, in the Year 1860, by Joseph L. Ross, in the Clerk's Office of the District Court of Massachusetts.

MATHEMATICAL BOOKS,

IMPORTED AND FOR SALE BY

SEVER AND FRANCIS, CAMBRIDGE,

SUCCESSORS TO JOHN BARTLETT.

ENGLISH.

American Ephemeris and Nautical Almanac. For the Years 1855–62. Each $1.50

BAKER. Treatise on Statics and Dynamics 0.30

BLAND. Algebraical Problems. 8vo. Second-hand 1.50

——— Key to do. 8vo. do. 1.75

——— Philosophical Problems. 8vo. do. 1.50

——— Geometrical Problems. 8vo. do. 1.50

BOOLE. Treatise on Differential Equations 4.00

BOOTH. Theory of Elliptic Integrals. 8vo. 2.25

BORDEN. System of Railway Formulæ. 8vo. 2.25

BRITISH NAUTICAL ALMANAC. 1861 and 1862, each 1.00

CARMICHAEL. Treatise on the Calculus of Operations. 8vo. . 2.75

CHAUVENET. Trigonometry. 8vo. 1.50

COURTENAY. Calculus. 8vo. 2.50

COOMBE. Solutions of the Cambridge Problems, 1840, 1841. 8vo. . 2.25

DEMORGAN. Differential and Integral Calculus 2.00

EARNSHAW. Dynamics. 8vo. 4.25

FISCHER. Logarithmic Tables of Seven Places. Translated from Bremiker's Vega. 8vo. Hf. mor. 2.50

FROST. Mathematical Questions of the Senate-House Examination Papers. 1838–1849. 8vo. 3.00

GAUSS. Theoria Motus. Translated by C. H. Davis. 4to. . . . 5.00

GREGORY. Differential and Integral Calculus by Walton. 8vo. . 5.50

HADDON. Differential Calculus 30

HAMILTON. Lectures on Quaternions. 8vo. 6.30

HANN. Integral Calculus 30

HEATHER. Descriptive Geometry 30

HEMMING. Differential and Integral Calculus. 8vo. 2.50

HUTTON. Mathematics 4.00

——— Recreations. 1 vol. 8vo. 4.00

——— Mathematical Tables. 8vo. 3.75

HYMER. Plane and Spherical Trigonometry 2.50

——— Algebraic Equations 3.00

——— Treatise on Conic Sections. 8vo. 2.75

——— Differential Equations and Calculus of Finite Differences. 8vo. 3.60

——— Integral Calculus 3.00

——— Geometry of Three Dimensions. 8vo. 3.00

JAMIESON. Solutions of the Senate-House Rider. 8vo. 2.25

LARDNER. System of Algebraic Geometry. 8vo. Second-hand . 1.75

MAIN and BROWN. Indicator and Dynometer 1.40

Mathematical Monthly. Edited by J. D. Runkle. Vol. I. In Nos. 3.00

Do. do. Hf. mor. 4.00

Mathematical Problems and Examples of the Senate-House Examination Papers. 1821–1836. With an Appendix containing the Senate-House Questions for 1837. 8vo. 3.00

PARKINSON. Elementary Mechanics. Post 8vo. 2.87

PEIRCE. Analytic Mechanics. 4to. 7.50

PEIRCE, J. M. Analytic Geometry. 8vo. 1.50

PHEAR. Elementary Hydrostatics. Post 8vo. 1.62

PRICE. Differential and Integral Calculus. 3 vols. 8vo. . . . 14.50

Quarterly Journal of Pure and Applied Mathematics. Sylvester, Ferriers, & Co. $6.00 per annum, *post-paid.*

RANKINE. Manual of Applied Mechanics 2.00

SALMON. Conic Sections 3.50

——— Higher Plane Curves 3.50

——— Higher Algebra 1.80

Solutions of the Cambridge Problems. 1848–1851. Ferrers and Jackson. 8vo. 4.50

——— 1854. Walton and McKenzie. 8vo. 2.50

——— 1857. Campion and Walton. 8vo. 2.50

——— Senate-House Riders. Jamieson. 8vo. 3.25

SPOTTISWOODE. Elementary Theorems relating to Determinants. 4to. 1.50

TATE. Exercises on Mechanics, with Key. 2 vols. 1.62

——— Mechanical Philosophy. 8vo. 3.25

——— Materials. 8vo. 1.75

TATE and STEEL. Dynamics. 8vo. 3.00

TODHUNTER. Analytical Statics. Post 8vo. 3.00

——— Differential and Integral Calculus. 2 vols. Post 8vo. 6.00

WALTON. Problems illustrative of Plane Coördinate Geometry. 8vo. 4.80

——— Mechanical Problems. 8vo. 5.50

——— Problems in Hydrostatics and Hydrodynamics. 8vo. . 3.00

——— On the Differential Calculus. 8vo. 3.25

WALTON and MCKENZIE. Solutions of the Cambridge Problems. 1854. 8vo. $3.25. Do. 1857. . . . . 2.50

——— Problems in Elementary Mechanics. 8vo. 3.00

WAUD. Algebraic Geometry. 8vo. Hf. cf. (Second-hand.) . . 1.50

WHEWELL. Analytic Statics. 8vo. 2.25

WILSON. Dynamics. 8vo. 2.87

WOOLHOUSE. Differential Calculus 30

FRENCH, &c.

AUDÉ. Poussée des Terres. 8vo. Hf. cf. $1.87

BELANGER. Résumé de Leçons de Géométrie Analytique et de Calcul Infinitésimal. 8vo. 2.75

BIOT. Traité des Equations differentielles. Hf. mor. 2.00

BOURCHARLAT. Eléments de Calcul Differential et du Calcul Integral. 8vo. 3.00

BONNET. Leçons de Mécanique Élémentaire à l'Usage des Candidates à l'Ecole Polytechnique. 8vo. 2.00

BRIOSCHE. Théorie des Determinants. 8vo. 2.00

BOURDON. Application de l'Algèbre à la Géométrie. 8vo. Hf. cf. 3.00

CALLET. Tables de Logarithms. 8vo. Hf. cf. 5.00

CHASLES. Traité de Géométrie Supérieure. 8vo. Hf. cf. . . . 6.00

CHOQUET. Traité élémentaire d'Algèbre. 8vo. Hf. cf. . . . 2.00

COURNOT. Traité élémentaire de la Theorie des Fonctions, et du Calcul Infinitésimal. 2 vols. 8vo. 4.00

CRELLE. Journal fur die reine und augewandte Mathematik. Per annum 4.00

DELISLE. Géométrie Analytique. 8vo. Hf. cf. 3.00

DUHAMEL. Cours d'Analyse Calcul Infinitésimal. 2 vols. 8vo. Hf. mor. 5.00

——— Cours de Mécanique. 2 vols. 8vo. Hf. mor. 4.50

DUPIN. Application de Géométrie et de Mécanique à la Marine aux Ponts et Chaussées. 4to. Hf. mor. 4.50

——— Developpments de Géométrie. 4to. Hf. mor. 4.50

——— Géométrie et Mécanique, appliquées aux Arts. 8vo. Hf. mor. 2.50

FRANCŒUR. Cours complet de Mathématiques pures. 2 vols. 8vo. 5.00

——— Éléments de Statique. 8vo. Hf. cf. 2.00

GAUSS. Méthode des Moindres Carrés. 8vo. Hf. cf. 1.62

KÖHLER. Logarithmisch Trigonometrisches Handbuch. 8vo. . 2.12

LACROIX. Traité du Calcul Differential et du Calcul Integral. 3 vols. in 4to, avec 18 pl. Mor. 35.00

LAGRANGE. Mécanique Analytique. 2 vols. 4to. 13.00

——— Théorie des Fonctions Analytiques. 4to. Hf. mor. . 6 00

——— Traité des Equations Numeriques de tous les Degrés. 4to. Hf. mor. 5.00

LALANDE. Tables de Logarithmes. 18mo. 1.25

LAPLACE. Système du Monde. 4to. Hf. mor. 5.50

LEGENDRE. Traité des Fonctions Elliptiques. 3 vols. 4to. Hf. mor. 20.00

LEROY. Traité de Stéréotomie comprenant les Applications de la Géométrie Descriptive à la Theorie des Ombres, la Perspective Lineaire, la Gnomonique, la Coupe des Pierres, et la Charpente. In 4to, avec Atlas de 74 pl. in fol. Paper, $9.00; hf. mor. . 12.00

——— Traité de Géométrie Descriptive. In 4to, avec Atlas de 71 pl. Paper, $3.00; hf. mor. $5.00 and 6.00

——— Analyse Appliquée à la Géométrie des trois Dimensions. 8vo. Hf. cf. 2.25

LIOUVILLE. Journal de Mathématiques Pures et Appliquées. 1836–1857. 22 vols. 3.00

MAHISTRE. Cours de Mécanique Appliquée. 8vo. 1.75

MONGE. Application de l'Analyse à la Géométrie. 4to. Hf. mor. 10.50

——— Traité élémentaire de Statique. 8vo. Hf. mor. . . . 2.00

MORIN. Aide-mémoire de Mécanique Pratique. 8vo. 6.00

MONTFERRIER. Encyclopédie Mathématique ou Exposition complète de Toutes les Branches des Mathématiques d'apres les Principles de la Philosophie des Mathématiques de Hoéné Wronski. 3 vols. 8vo. To be continued in monthly Livraisons. 5.00

NAVIER. (Ketten) Hängbrüchen. 4to. 3.75

OLIVIER. Théorie Géométrique des Engrenages. 4to. Hf. mor. . 9.00

——— Developpements de Géométrie Descriptive. 2 vols. 4to. Hf. mor. 9.00

——— Complément de Géométrie Descriptive. 2 vols. 4to. 9.00

——— Mémoires de Géométrie Descriptive, Theorique, et Appliquée. 2 vols. 4to. Hf. mor. 10.00

——— Application de la Géométrie Descriptive. 2 vols. 4to. 10.00

——— Cours de Géometrie Descriptive. 2 vols. 4to. Hf. mor. 10.00

POINSOT. Eléments de Statique. 8vo. Hf. cf. 2.50

POISSON. Mécanique. 2 vols. 8vo. 6.00

PONCELET et LESBROS. Experiences Hydrauliques. 4to. Hf. mor. 4.00

PONTECOULANT. Système du Monde. 4 vols. 8vo. Hf. cf. . 16.00

POUILLET. Eléments de Physique Experimentale. 3 vols. 8vo. Hf. cf. 7.50

SERRET. Cours d'Algèbre supérieure. 8vo. Hf. cf. 3.50

——— Eléments d'Arithmétique. 8vo. Hf. cf. 1.50

TERQUEM. Nouvelles Annales de Mathématiques. Monthly. Per annum 4.00

——— Nouvelles Annales de Mathématiques, 1856–7. 2 vols. Hf. cf. 6.50

——— Nouvelles Annales de Mathématiques. Tomes VIII. à XVII. Corresponding to the Years 1849–1859. 10 vols. Hf. cf. 16.00

——— Bulletin Bibliographie, d'Histoire, et de Biographie Mathématiques. 8vo. Hf. cf. 3.00

TRESCA. Géométrie Descriptive. 8vo. Hf. mor. 3.00

VERHULST. Traité élémentaire des Fonctions Elliptiques. 8vo. Hf. cf. 3.50

VIEILLE. Cours complémentaire d'Analyse et de Mécanique. 8vo Hf. cf. 2.50

SCIENTIFIC BOOKS.

RECENTLY RECEIVED.

AIRY. On the Algebraical and Numerical Theory of Errors of Observations and the Combination of Observations. 12mo. . $2.00
DE MORGAN. Essay on Probabilities. 12mo. 1.50
DREW. A Geometrical Treatise on Conic Sections. 12mo. . . 1.50
BROOK SMITH. Arithmetic in Theory and Practice for Advanced Pupils. Part I. 1.50
BRESSE. Cours de Mécanique Appliquée. Seconde Partie. Hydraulique. 8vo. paper. 2.25
COMBEROUSSE. Cours de Mathématiques à l'Usage des Candidets à l'École Centrale des Arts et Manufactures. Tome premier. Arithmétique. Algébre Élémentaire. 8vo. paper. . . . 2.25
HATON DE LA GOUPILLIERE. Elements du Calcul Infinitesimal. 8vo. paper. 1.75
LACROIX. Traité Élémentaire de Calcul Différentiel et de Calcul Intégral. Sixième edition. Tome premier. 8vo. paper. . . 4.00
STURM. Cours de Mécanique de l'École Polytechnique. Tome premier. 8vo. paper. 3.50
TODHUNTER. A History of the Progress of the Calculus of Variations during the Nineteenth Century. 8vo. 4.00
——— Algebra 2.25
——— Plane Trigonometry 1.50
——— Spherical Trigonometry 1.25
——— Analytical Geometry, Examples of 1.25
——— Conic Sections 3.00

SECOND-HAND MATHEMATICAL BOOKS.

AIRY. Trigonometry. Tides and Waves. Figure of Earth. 4to. $4.50
BAGAY. Nouvelles Tables Astronomiques et Hydrographiques. 4to. hf. cf. *Scarce* 20.00
BLAND. Algebraical Problems. 8vo. 1.75
——— Key to do. 8vo. 1.75
——— Philosophical Problems. 8vo. 1.50
——— Geometrical Problems. 8vo. 1.75
——— Elements of Hydrostatics 1.50
DE MORGAN. Essay on Probabilities 1.00
——— The Globes, Celestial and Terrestrial 1.25
Diarian Miscellany, Mathematical and Poetical, by Hutton. 5 vols. 12mo. cf. 4.00
Cambridge Problems. From 1801 to 1820. 8vo. 1.00
EARNSHAW. Dynamics. 8vo. 2.50
GRANT. Plane Astronomy. 8vo. 1.25
GREGORY AND WALTON. Application of Analysis to Solid Geometry. 8vo. 3.00
HERSCHEL. Calculus of Finite Differences. 8vo. 4.00
HYMER. Conic Sections. 3d ed. 8vo. 2.00
——— Analytical Geometry of Three Dimensions. 1st ed. 8vo. 1.00
——— do. do. 2d ed. 8vo. 1.50
——— Plane and Spherical Trigonometry. 2d ed. 8vo. . . 1.25
——— do. do. 3d ed. 8vo. . . 1.50
——— Integral Calculus. 3d ed. 8vo. hf. cf. 4.00
——— Astronomy. 2d ed. 8vo. 2.00
LACROIX. Elements of Algebra, translated by Prof. John Farrar. 5th ed. 8vo. $1.50
Ladies' Diary, Poetical and Mathematical. 1784-1790. 16mo. . 1.00
LAPLACE. Elementary Illustrations of Celestial Mechanics. 8vo. . 1.25
——— Analytical Mechanics, Translated by Rev. John Tophs 1.00
LARDNER. System of Algebraic Geometry. 8vo. 1.75
——— Elements of Euclid. 5th Edition. 8vo. 1.50
——— Differential and Integral Calculus. 8vo. 2.00
LEYBOURN. Mathematical Questions with Solutions. 4 vols. 8vo. 8.00
——— Mathematical Institutions. 4to. 1.50
Mathematical Register; or, Diarian Repository. 4to. 2.50
Mathematical Companion, The Gentleman's, for the Years 1810-11-12-14-15-17-18-19-20-24-25. 16mo. Unbound. 11 Nos. . 2.00
MOIGNO. Leçons de Calcul Differential. Vol. I. Hf. cf. Scarce. 9.00
MOSELEY. Hydrostatics and Hydrodynamics. 8vo. Scarce. . . 4.00
O'BRIEN. Mathematical Tracts. Part I. 0.75
PEACOCK. Algebra 2.00
POISSON. Traité de Mécanique. 2 vols. 8vo. hf. bd. 5.00
PRICE. Differential Calculus. Vol. I. 1st ed. 1.50
PUCKLE. Conic Sections. 1st ed. 12mo. 1.50
STEVENSON. Algebraic Equations. 8vo. 1.00
TODHUNTER. Analytical Statics. 12mo. 2.00
WAUD. Algebraic Geometry. 8vo. Hf. cf. 1.50
YOUNG. Algebraical Equations. 12mo. 1.00

☞ Orders for the Importation of English, French, and German Books will be promptly executed, and at moderate prices. Early copies of all the new mathematical books published in England and France will be received as soon as published.

BOOKS FOR PUBLIC LIBRARIES IMPORTED FREE OF DUTY.

TEXT-BOOKS USED IN HARVARD COLLEGE.

THE ORATION OF ÆSCHINES AGAINST CTESIPHON. With Notes, by PROFESSOR J. T. CHAMPLIN. 12mo. cloth. $0.88
THE CLOUDS OF ARISTOPHANES. With Notes, by PROFESSOR C. C. FELTON. 12mo. Cloth 1.00
THE BIRDS OF ARISTOPHANES. With Notes, by PROFESSOR C. C. FELTON. 12mo. cloth 1.00
THE PANEGYRICUS OF ISOCRATES. With Notes, by PROFESSOR C. C. FELTON. 12mo. Cloth 75
SELECTIONS FROM THE GREEK HISTORIANS. With Notes, by PROFESSOR C. C. FELTON. 12mo. half mor. . . . 1.50
SELECTIONS FROM MODERN GREEK WRITERS, IN PROSE AND POETRY. With Notes, by PROFESSOR C. C. FELTON. 12mo. Cloth 1.00
THE AIAS OF SOPHOKLES. With Critical and Explanatory Notes, by J. B. M. GRAY. 12mo. cloth 1.25
PRONUNCIATION AND HISTORY OF THE GREEK ALPHABET. By E. A. SOPHOCLES, Professor in Harvard College. 12mo. Cloth 1.00
CICERO. BRUTUS. Edited, with Notes, by PROFESSOR C. BECK. 16mo. Cloth 75
CICERO. Immortality of the Soul; The Tusculan Disputations, Book First; The Dream of Scipio; and Extracts from the Dialogues on Old Age and Friendship. With English Notes, by THOMAS CHASE, Tutor in Harvard College. 16mo. Cloth. . .75
CICERO. PRO CLUENTIO. With English Notes, by AUSTIN STICKNEY, M. A., Professor of Latin in Trinity College. 12mo. Cloth 75
CONSTITUTIONAL DOCUMENTS OF ENGLAND AND AMERICA, from Magna Charta to the Federal Constitution of 1789. Compiled and Revised, with Notes, by F. BOWEN, Professor of Moral Philosophy in Harvard College. 8vo. Cloth . . . $0.88
HODGES, R. M. PRACTICAL DISSECTIONS. By RICHARD M. HODGES, M. D., Demonstrator of Anatomy in the Medical Department of Harvard College. 16mo. 1.00
HORACE. With Notes, by A. J. MACLEANE, M. A. Revised and Edited by R. H. CHASE, A. M. With Introduction to the Metres, by PROFESSOR CHARLES BECK. Revised edition. 12mo. half mor. 1.25
LATHAM'S ELEMENTARY ENGLISH GRAMMAR. Revised by PROFESSOR CHILD. With an Appendix, by PRESIDENT GOODWIN, Trinity College. 16mo. Cloth 75
PEIRCE, J. M. ANALYTIC GEOMETRY. 8vo. 1.50
ROELKER, B. A GERMAN READER FOR BEGINNERS. Compiled by BERNARD ROELKER, A. M., Instructor in Harvard University. 12mo. Cloth 1.00
SYNTAX OF THE MOODS AND TENSES OF THE GREEK VERB. By W. W. GOODWIN, Ph. D., Prof. in Harvard College. 12mo. Cloth 1.25
VERNON, EDWARD JOHNSTON, B. A. GUIDE TO THE ANGLO-SAXON TONGUE. 12mo Cloth 1.25
WHATELY. LESSONS ON MORALS AND CHRISTIAN EVIDENCES. Edited by REV. F. D. HUNTINGTON. 16mo. . . .75

PUBLISHED AND FOR SALE BY

SEVER & FRANCIS,

Booksellers to the University, Cambridge.

THE NEW AMERICAN CYCLOPÆDIA.

D. APPLETON & CO.,

Nos. 443 & 445 Broadway, New York,

HAVE JUST PUBLISHED

Vol. XI. of the

New American Cyclopædia,

A POPULAR DICTIONARY OF GENERAL KNOWLEDGE,

EDITED BY

GEORGE RIPLEY and CHARLES A. DANA,

Aided by a Numerous Select Corps of Writers from all parts of the United States.

This work is being published in about 16 large octavo volumes, each containing 750 two-column pages.

VOLS. I. TO XI. NOW READY.

To be followed by a new Volume every three months.

Price — In cloth, $3; sheep, $3.50; half morocco, $4; half Russia, $4.50 each.

From De Bow's Review, New Orleans.

"This great work, published by the Appletons of New York, has reached its eighth volume. It is one of the most valuable Cyclopædias in the world, and for Americans the most valuable. It is contributed to by many of the ablest and best writers of the South, and is as fair to every section of the Union as it is possible for a work to be. It will be completed in about seven other volumes."

The following notice in *Silliman's American Journal of Sciences and Arts*, for March, is from the pen of the venerable senior editor, one of the highest living authorities on the subject: —

"In the American Cyclopædia the articles on Science are numerous and valuable, and elevate the work to the character of a compendium of modern science. These articles are in most cases written with decided ability, and evidently by persons who are familiar with the topics they discuss. While many of the less important subjects are presented with luminous brevity, others are more fully expanded. Among these are many topics of Natural History. Chemistry is presented with its equivalence and laws of combination, illustrated by many of its modern discoveries and practical applications; of the latter, an example is found in the full account of the manufacture of gelatine, of beer, and bricks, and in the ample history of gas-lighting. Neology, voltaic electricity, magnetism, and other departments of pure or applied science, are treated with reasonable fulness. As a literary work the Cyclopædia is written in a pure and chaste style, and exhibits the candor and fairness which should ever adorn a record of universal knowledge. L. S."

OUR SUBSCRIPTION LIST

Already numbers TWELVE THOUSAND names, from every section of the country, which evidences the universal popularity of this NATIONAL WORK.

THE NEW AMERICAN CYCLOPÆDIA

Is sold exclusively by subscription, and Agents have been appointed for almost all sections of the Union. In case, however, Agents are not to be found, we will receive subscriptions and forward copies of the work, free of expense, on receipt of the price. Persons at a distance may send money for one or more volumes at a time, it not being necessary to take the whole eight volumes at once, unless they feel inclined to do so. Those living in the city will find a subscription-book at our store, where their names will be entered, and the volumes, as published, sent to any part of the city.

WARREN'S

Descriptive Geometry.

ORTHOGRAPHIC PROJECTIONS.

BOOK I. — General Problems.

BOOK II. — Applications to Isometrical Projections, Spherical Trigonometry, Topographical Projections, and Graphic Transformations.

8vo. 447 pp., and 36 sheets of plates. Price, $3.00.

BY S. EDWARD WARREN, C. E.,

PROFESSOR OF DESCRIPTIVE GEOMETRY AND GEOMETRICAL DRAWING IN THE RENSSELAER POLYTECHNIC INSTITUTE, TROY, N. Y.

This work has been prepared as the result of a number of years' experience in teaching the interesting and useful branch of mathematical science of which it treats, in the Rensselaer Polytechnic Institute.

It has been warmly welcomed and commended in discriminating and independent critical notices, some of which are appended.

The study of Descriptive Geometry, as a *means of intellectual training*, as well as an *aid in important industrial pursuits*, has peculiar merits, and the attention of professors, teachers, and students of Pure or Applied Mathematics in the scientific schools, colleges, &c., and members of the Engineering profession, are invited to an examination of the above work.

RECOMMENDATIONS.

"Professor Warren's book is far more comprehensive than any work on Descriptive Geometry before offered to American, or, so far as we know, to English readers. . . . It has many positive merits." — *Mathematical Monthly, February*, 1861.

From WM. MITCHELL GILLESPIE, LL. D., *Prof. Civil Engineering, Union College, N. Y.*

"I have looked over your work on Descriptive Geometry with surprise and delight, — surprise that the most comprehensive book on that subject in the English language should be an American one, and delight that it should be so good a one, &c., &c. I shall recommend it to my mathematical friends."

From J. S. WOODMAN, *Prof. Civil Engineering, Dartmouth College, N. H.*

"I have hastily examined your excellent treatise on the 'Orthographic Projections of Descriptive Geometry,' and like it much. The parts of Descriptive Geometry you have undertaken are so systematized and so fully discussed, that your book must have an important influence in bringing that science into public notice and favor."

JOHN WILEY, Publisher and Importer,

56 WALKER STREET, NEW YORK.

☞ In press, by the same author, "A Manual of Elementary Geometrical Drawing, involving three Dimensions." With numerous plates. 1 vol., 12mo. Price, $1.25.

RECENT IMPORTANT WORKS.

Annual of Scientific Discovery for 1861; or, YEAR-BOOK OF FACTS IN SCIENCE AND ART, exhibiting the most important Discoveries and Improvements in Mechanics, Useful Arts, Natural Philosophy, Chemistry, Astronomy, Meteorology, Zoölogy, Botany, Mineralogy, Geology, Geography, Antiquities, &c. Edited by David A. Wells, A. M. 12mo. Cloth, $ 1.25. (*Just published.*)

No other single publication, American or foreign, presents such a mass of fresh, varied, interesting, and valuable information, adapted to the wants of persons in almost every walk of life; and its acquisition has become a necessity to all who would acquaint themselves with the progress of discovery with each successive year.

The Romance of Natural History. By Philip Henry Gosse. With numerous Elegant Illustrations. 12mo. Cloth, $ 1.25. (*Just published.*)

One of the most attractive issues of the season. Its charming descriptions and admirable illustrations are giving it a marked popularity. The first edition was exhausted in a few days.

The Pulpit of the American Revolution; Or, The Political Sermons of the Period of 1776. With a HISTORICAL INTRODUCTION, NOTES, and ILLUSTRATIONS. By J. Wingate Thornton, A. M. 12mo. Cloth, $ 1.25. (*Just published.*)

"This work contains the sum and substance of the whole history of the American Revolution."

The Puritans; or the Court, Church, and Parliament of England, during the reigns of Edward VI. and Elizabeth. By Samuel Hopkins, Author of "Lessons at the Cross," etc. In 3 vols. Vols. I. and II. *now ready.* 8vo. Cloth, $ 2.50 per vol.

This is a masterly work, by one of the most distinguished of living authors.

Essays in Biography and Criticism. By Peter Bayne, M. A., Author of "The Christian Life, Social and Individual." Arranged in two Series or Parts. 12mo. Cloth, $ 1.25 each.

"This is a book of rare literary excellence. Its analytical powers, its earnestness of purpose, and its wealth of language, are alike remarkable." —*Mrs. Sigourney.*

The Christian Life; SOCIAL AND INDIVIDUAL. By Peter Bayne, M. A. 12mo. Cloth, $ 1.25.

"It is full of noble thought and brilliant illustration." — *Hon. R. C. Winthrop.*

Thesaurus of English Words and Phrases; so classified and arranged as to facilitate the expression of ideas, and assist in literary composition. By Peter Mark Roget, late Secretary of the Royal Society, London, etc. Revised and edited, with a List of Foreign Words defined in English, and other additions, by Barnas Sears, D. D., President of Brown University. A New American Edition, with Additions and Improvements. 12mo. Cloth, $ 1.50.

"A work of great utility. It will give a writer the word he wants, when that word is on the tip of his tongue, but altogether beyond his reach." — *N. Y. Times.*

British Novelists and their Styles. Being a Critical Sketch of the History of British Prose Fiction. By David Masson, M. A., author of "The Life and Times of John Milton," etc., etc. 16mo. Cloth, 75 cents.

"This book is a hearty, honest, and learned history of the British novel, — remarkable for its acute and philosophical criticism, and the classic purity and elegance of its style." — *Providence Journal.*

The Preacher and the King; or, Bourdaloue at the Court of Louis XIV. By L. F. Bungener. With a Likeness and Biographical Sketch of the Author. 12mo. Cloth, $ 1.25.

The Priest and the Huguenot; or, Persecution in the Age of Louis XV. By L. F. Bungener, Author of "The Preacher and the King." 2 vols. 12mo. Cloth, $ 2.25.

Plurality of Worlds. With an Introduction by Edward Hitchcock, D. D., and a Supplementary Dialogue, in which the author's reviewers are reviewed. 12mo. Cloth, $ 1.

☞ The above works are worthy of special attention. Bungener is a writer of great originality and power; and "The Plurality of Worlds" is one of the marked productions of modern times.

Geographical Studies. Translated from the German of Carl Ritter. By Rev. W. L. Gage. With a Sketch of the Author's Life and a Portrait. 12mo. Cloth, $ 1. (*Ready in a few days.*)

This volume contains the grand generalizations of Ritter's Erdkünde, his Lectures on the relations of Geography and History, and important papers on Physical Geography.

Mental Philosophy; Including the Intellect, Sensibilities, and Will. By Joseph Haven, Professor of Intellectual and Moral Philosophy, Amherst College. Royal 12mo. Cloth, $ 1.50.

Prof. Park, of Andover, says: "It is DISTINGUISHED for its clearness of style, perspicuity of method, candor of spirit, acumen, and comprehensiveness of thought. I have been heartily interested in it."

"Earnestness, candor, and accuracy and elegance of diction characterize every page, and render the work as interesting to the general reader as it is valuable to the scholar." — D. H. COCHRAN, *Principal and Prof. of Moral and Intellectual Philosophy, State Normal School, Albany.*

This work has been extensively adopted as a text-book in Colleges and High Schools throughout the country.

Moral Philosophy; Including Theoretical and Practical Ethics. By Joseph Haven, D. D., late Professor of Moral and Intellectual Philosophy in Amherst College; Author of "Mental Philosophy." Royal 12mo. Cloth, $ 1.25.

It is scientific in method and thorough in discussion, and its views on unsettled questions in morals are discriminating and sound. It treats largely of Political ethics, — a department of morals of great importance to American youth, but generally overlooked in text-books. In the history of ethical opinions, it is unusually rich and elaborate.

Lectures on Logic. By Sir William Hamilton, Bart., Professor of Logic and Metaphysics in the University of Edinburgh. With Notes from original materials, and an Appendix containing the latest development of his new Logical Theory. Edited by H. Longueville Mansel, B. D., Oxford, and John Veitch, A. M., Edinburgh. Royal 8vo. Cloth, $ 3.00.

This volume completes the Metaphysical and Logical series of Sir William Hamilton, and will enable American scholars to possess themselves of the invaluable works of this eminent author, in two elegant octavo volumes, at six dollars, — half the price of the English edition.

Lectures on Metaphysics. By Sir William Hamilton, Bart. Edited by Prof. H. L. Mansel, B. D., Oxford, and John Veitch, M. A., Edinburgh. Royal 8vo. Cloth, $ 3.00.

Hamilton's Lectures on Metaphysics have been received with eminent favor and already adopted as a text-book in Harvard, Yale, Princeton, and other distinguished Colleges.

Elements of Moral Science. By Francis Wayland, D. D., late President of Brown University. 12mo. Cloth, $ 1.25.

Elements of Political Economy. By Francis Wayland, D. D. 12mo. Cloth, $ 1.25.

The above works by Dr. Wayland are used as text-books in many of the colleges and higher schools throughout the Union, and are highly approved.

The Elements of Geology; Adapted to Schools and Colleges. With numerous Illustrations. By J. R. Loomis, President of Lewisburg University, Pa. 12mo. Cloth, 75 cents.

"It is surpassed by no work before the American public." — M. B. ANDERSON, LL. D., *President of Rochester University.*

Principles of Zoology; Touching the Structure, Development, Distribution, and Natural Arrangement of the RACES OF ANIMALS, Living and Extinct; with numerous Illustrations. For the use of Schools and Colleges. Part I. COMPARATIVE PHYSIOLOGY. By Louis Agassiz and Augustus A. Gould. Revised edition. $ 1.00.

The Earth and Man; Lectures on Comparative Physical Geography, in its relation to the History of Mankind. By Arnold Guyot. With Illustrations. 12mo. Cloth, $ 1.25.

Prof. Louis Agassiz, of Harvard University, says: "It will not only render the study of Geography more attractive, but actually show it in its true light."

Hugh Miller's Works. Seven volumes, uniform style, in an elegant box. Embossed Cloth, $ 8.25.

GOULD & LINCOLN, BOSTON.